This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

**Usage guidelines**

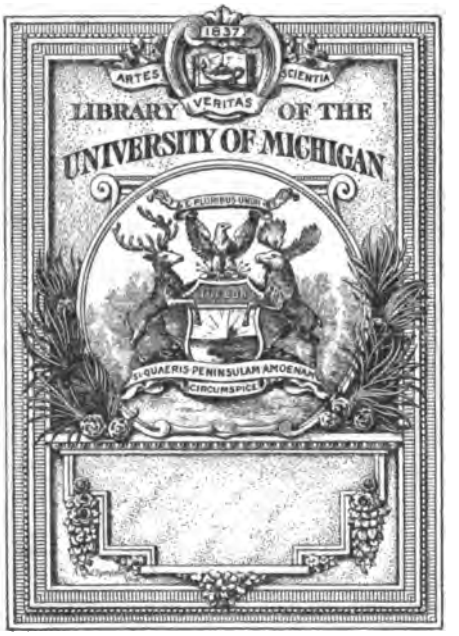
Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

**About Google Book Search**

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`


1837
ARTES
VERITAS
SCIENTIA
LIBRARY OF THE
UNIVERSITY OF MICHIGAN
E PLURIBUS UNUM
TUEBOR
SI QUAERIS PENINSULAM AMOENAM
CIRCUMSPICE

Q
41
.C183

# TRANSACTIONS

OF THE

# CAMBRIDGE

# PHILOSOPHICAL SOCIETY.

---

VOLUME XVII.

CAMBRIDGE:
AT THE UNIVERSITY PRESS

M.DCCC.XCIX.

## ADVERTISEMENT.

---

*The Society as a body is not to be considered responsible for any facts and opinions advanced in the several Papers, which must rest entirely on the credit of their respective Authors.*

---

The Society takes this opportunity of expressing its grateful acknowledgments to the Syndics of the University Press for their liberality in taking upon themselves the expense of printing this Part of the Transactions.

# CONTENTS.


PAGE

I. *Theorems relating to the Product of two Hypergeometric Series.* By Prof. W. McF. Orr, M.A., Royal College of Science, Dublin ..... 1

II. *On the possibility of deducing magneto-optic phenomena from a direct modification of an electro-dynamic energy function.* By J. G. Leathem, M.A., Fellow of St John's College ..... 16

III. *On the solutions of the equation* $(V^2 + \kappa^2)\psi = 0$ *in elliptic coordinates and their physical applications.* By R. C. Maclaurin, St John's College ..... 41

IV. *Certain Systems of Quadratic Complex Numbers.* By A. E. Western, B.A., Trinity College, Cambridge. ..... 109

V. *Partitions of Numbers whose Graphs possess Symmetry.* By Major P. A. MacMahon, R.A., D.Sc., F.R.S., Hon. Mem. C.P.S. ..... 149

VI. *On Divergent (or Semiconvergent) Hypergeometric Series.* By Professor W. McF. Orr, M.A., Royal College of Science, Dublin ..... 171

VII. *A semi-inverse method of solution of the equations of elasticity, and its application to certain cases of æolotropic ellipsoids and cylinders.* By C. Chree, Sc.D., F.R.S. ... 201

VIII. *On the Change of a System of Independent Variables.* By E. G. Gallop, M.A., Fellow of Gonville and Caius College..... 231

IX. *On Divergent Hypergeometric Series.* By Professor W. McF. Orr, M.A. ..... 283


137485

# TRANSACTIONS

OF THE

# CAMBRIDGE

# PHILOSOPHICAL SOCIETY.

---

VOLUME XVII. PART I.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.

M.DCCC.XCVIII.

## ADVERTISEMENT.

---

*The Society as a body is not to be considered responsible for any facts and opinions advanced in the several Papers, which must rest entirely on the credit of their respective Authors.*

---

The Society takes this opportunity of expressing its grateful acknowledgments to the Syndics of the University Press for their liberality in taking upon themselves the expense of printing this Part of the Transactions.

# CONTENTS.

PAGE

I. *Theorems relating to the Product of two Hypergeometric Series.* By Prof. W. McF. Orr, M.A., Royal College of Science, Dublin ..... 1

II. *On the possibility of deducing magneto-optic phenomena from a direct modification of an electro-dynamic energy function.* By J. G. Leathem, M.A., Fellow of St John's College ..... 16

III. *On the solutions of the equation* $(V^2 + \kappa^2)\psi = 0$ *in elliptic coordinates and their physical applications.* By R. C. Maclaurin, St John's College ..... 41

# I. *Theorems relating to the Product of two Hypergeometric Series.* By Prof. W. McF. Orr, M.A., Royal College of Science, Dublin.

[Received June 1897.]

1. The following theorem is stated without proof by Cayley (*Phil. Mag.* Nov. 1858, and *Collected Papers*, Vol. III. page 268), viz., writing as usual

$$F(\alpha,\ \beta;\ \gamma;\ x) = 1 + \frac{\alpha\,.\,\beta}{1\,.\,\gamma}x + \frac{\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta+1}{1\,.\,2\,.\,\gamma\,.\,\gamma+1}x^2 + \ldots\ldots,$$

then the product $\quad F(\alpha,\ \beta;\ \gamma+\tfrac{1}{2};\ x)\,.\,F(\gamma-\alpha,\ \gamma-\beta;\ \gamma+\tfrac{1}{2};\ x)$

is connected with $\quad (1-x)^{\alpha+\beta-\gamma}\,.\,F(2\alpha,\ 2\beta;\ 2\gamma;\ x)$

by a simple relation; for if the last-mentioned expression is put equal to

$$1 + Bx + Cx^2 + Dx^3 + \ldots,$$

then the product in question is equal to

$$1 + \frac{\gamma}{\gamma+\frac{1}{2}}\,.\,Bx + \frac{\gamma\,.\,\gamma+1}{\gamma+\frac{1}{2}\,.\,\gamma+\frac{3}{2}}\,.\,Cx^2 + \frac{\gamma\,.\,\gamma+1\,.\,\gamma+2}{\gamma+\frac{1}{2}\,.\,\gamma+\frac{3}{2}\,.\,\gamma+\frac{5}{2}}\,.\,Dx^3 + \ldots.$$

The object of this paper is to establish the above and other similar theorems.

2. Having given any series $\quad u \equiv \Sigma a_r x^r$ .....(1),

if we form from it another $\quad v \equiv \Sigma b_r x^r$ .....(2),

by means of the relation $\quad \dfrac{b_{r+1}}{b_r} = \dfrac{r+\theta}{r+\phi}\,.\,\dfrac{a_{r+1}}{a_r}$ .....(3),

where $\theta$ and $\phi$ are constants we shall express the connection between $v$ and $u$ by writing

$$v = (\theta;\ \phi;\ u) \text{ or } v = \{\theta;\ \phi;\ u\} \quad \ldots\ldots\ldots\ldots(4).$$

The method of proof pursued is to take the normal forms of the equations satisfied by two independent hypergeometric functions, obtain the linear differential equation satisfied by the product of the solutions of these, and investigate in what cases it can be identical with that satisfied by

$$y = (x-1)^{-\kappa}x^{-h}\{\theta;\ \phi;\ (1-x)^{-\sigma}z\} \ldots\ldots\ldots\ldots(5),$$

where $z$ is a third hypergeometric function, all three functions having $x$ as argument.

3. Considering any two equations of the types

$$(D^2+I)\,\zeta=0,\quad (D^2+I')\,\zeta'=0 \quad\text{.....(6), (7),}$$

it is easily found that the product $\zeta\zeta'$ satisfies the equation

$$D\left[\frac{D^2y+2PDy+yDP}{Q}\right]+Qy=0 \quad\text{.....(8),}$$

where $I+I'=P$, $I-I'=Q$; $D$ denoting differentiation with respect to $x$. (Compare Schafheitlin, *Pr.* (No. 99), *Sophien-Realgymn.* Berlin, 29 S. 4°.)

If (6), (7) are the normal forms of the equations satisfied by

$$\eta=F(\alpha,\beta;\gamma;x) \text{ and } \eta'=F(\alpha',\beta';\gamma';x)$$

respectively, we have $\quad I=\frac{1}{4}\left[\frac{1-\lambda^2}{x^2}+\frac{\lambda^2-\mu^2+\nu^2-1}{x(x-1)}+\frac{1-\nu^2}{(x-1)^2}\right],$

where $\quad \lambda=1-\gamma,\quad \mu=\alpha-\beta,\quad \nu=\gamma-\alpha-\beta,$

with corresponding relations in case of the dashed letters.

The most general solution of (8) in this case then is

$$y=x^{\frac{1}{2}(\gamma+\gamma')}(1-x)^{\frac{1}{2}(\alpha+\beta-\gamma+\alpha'+\beta'-\gamma'+2)}\{A\eta_1\eta_1'+B\eta_1\eta_2'+C\eta_2\eta_1'+D\eta_2\eta_2'\} \quad\text{.....(9),}$$

where $\eta_1$, $\eta_2$ are any independent solutions of the equation satisfied by $\eta$; $\eta_1'$, $\eta_2'$ are any independent solutions of that satisfied by $\eta'$, and $A$, $B$, $C$, $D$ are arbitrary constants.

4. We now proceed to find the equation whose solution is given by (5) where $z$ is the most general solution of

$$x(1-x)\frac{d^2z}{dx^2}+(A-Bx)\frac{dz}{dx}-Cz=0 \quad\text{.....(10).}$$

Writing $(1-x)^{-\sigma}z=u$, we have

$$\begin{aligned}x^2D^2u+(B+2\sigma)\,x^2Du+\{\sigma(\sigma-1)+\sigma B+C\}\,xu\\ -\{2x^2D^2u+(A+B+2\sigma)\,xDu+(\sigma A+C)\,u\}+xD^2u+ADu=0 \quad\text{.....(11).}\end{aligned}$$

If a solution of this be $\qquad u=\Sigma a_r x^r,$

we obtain the relation

$$\begin{aligned}\{(r-1)(r-2)+(B+2\sigma)(r-1)+\sigma(\sigma-1)+\sigma B+C\}\,a_{r-1}\\ -\{2r(r-1)+(A+B+2\sigma)\,r+\sigma A+C\}\,a_r+(r+1)(r+A)\,a_{r+1}=0 \quad\text{.....(12).}\end{aligned}$$

Making the transformation indicated in (2), (3), and writing $D'$ for $x\frac{d}{dx}$, we obtain the equation

$$\begin{aligned}x(D'+\theta+1)(D'+\theta)\{D'(D'-1)+(B+2\sigma)\,D'+\sigma(\sigma-1)+\sigma B+C\}\,v\\ -(D'+\theta)(D'+\phi-1)\{2D'(D'-1)+(A+B+2\sigma)\,D'+\sigma A+C\}\,v\\ +\frac{1}{x}(D'+\phi-1)(D'+\phi-2)\,D'(D'+A-1)\,v=0 \quad\text{.....(13).}\end{aligned}$$

On writing $y' = x^{-h}v$ the equation for $y'$ may be obtained from (13) by changing $D'$ into $D'+h$, and on writing

$$\left.\begin{aligned}
&A+2h=A'\\
&B+2\sigma+2h=B'\\
&h(h+A-1)=L\\
&2h(h-1)+h(A+B+2\sigma)+\sigma A+C=M\\
&h(h-1)+h(B+2\sigma)+\sigma(\sigma-1)+\sigma B+C=N\\
&\theta+h+1=\theta'\\
&\phi+h+1=\phi'
\end{aligned}\right\}\quad\text{.....(14)},$$

$$\left.\begin{aligned}
&2\phi'+A'=P_1\\
&2\theta'+B'-2\phi'-A'+4=Q_1\\
&2(\phi'-1)A'+\phi'(\phi'-1)+L=P_2\\
&2\phi'(\theta'-\phi'+2)+(\theta'+\phi')(A'+B')-4(\phi'-1)A'+M-2L=Q_2\\
&(\theta'-\phi'+2)(\theta'-\phi'+1)+(\theta'-\phi'+2)(B'-A')+L-M+N=R_2\\
&(\phi'-2)\{(\phi'-1)A'+2L\}=P_3\\
&(\phi'-1)\{\theta'(A'+B')-2(\phi'-2)A'\}+(\theta'+\phi'-2)M-4(\phi'-2)L=Q_3\\
&\theta'(\theta'-\phi'+2)B'-(\theta'-\phi'+2)(\phi'-1)A'+2(\phi'-2)L-(\theta'+\phi'-2)M+2\theta'N=R_3\\
&(\phi'-2)(\phi'-3)L=P_4\\
&(\phi'-2)\{(\theta'-1)M-2(\phi'-3)L\}=Q_4\\
&\theta'(\theta'-1)N-(\theta'-1)(\phi'-2)M+(\phi'-2)(\phi'-3)L=R_4
\end{aligned}\right\}\quad\text{...(15)},$$

and rearranging, this equation becomes

$$\left[D^4+\left\{\frac{P_1}{x}+\frac{Q_1}{x-1}\right\}D^3+\left\{\frac{P_2}{x^2}+\frac{Q_2}{x(x-1)}+\frac{R_2}{(x-1)^2}\right\}D^2\right.$$
$$\left.+\left\{\frac{P_3}{x^3}+\frac{Q_3}{x^2(x-1)}+\frac{R_3}{x(x-1)^2}\right\}D+\left\{\frac{P_4}{x^4}+\frac{Q_4}{x^3(x-1)}+\frac{R_4}{x^2(x-1)^2}\right\}\right]y'=0\text{.....(16)}.$$

If we now write $$y=(x-1)^{-\kappa}y',$$

we finally obtain as the equation for $y$,

$$D^4y+\left\{\frac{P_1}{x}+\frac{Q_1+4\kappa}{x-1}\right\}D^3y+\left\{\frac{P_2}{x^2}+\frac{Q_2+3\kappa P_1}{x(x-1)}+\frac{R_2+3\kappa Q_1+6\kappa(\kappa-1)}{(x-1)^2}\right\}D^2y$$
$$+\left\{\frac{P_3}{x^3}+\frac{Q_3+2\kappa P_2}{x^2(x-1)}+\frac{R_3+2\kappa Q_2+3\kappa(\kappa-1)P_1}{x(x-1)^2}+\frac{2\kappa R_2+3\kappa(\kappa-1)Q_1+4\kappa(\kappa-1)(\kappa-2)}{(x-1)^3}\right\}Dy$$
$$+\left\{\frac{P_4}{x^4}+\frac{Q_4+\kappa P_3}{x^3(x-1)}+\frac{R_4+\kappa Q_3+\kappa(\kappa-1)P_2}{x^2(x-1)^2}+\frac{\kappa R_3+\kappa(\kappa-1)Q_2+\kappa(\kappa-1)(\kappa-2)P_1}{x(x-1)^3}\right.$$
$$\left.+\frac{\kappa(\kappa-1)R_2+\kappa(\kappa-1)(\kappa-2)Q_1+\kappa(\kappa-1)(\kappa-2)(\kappa-3)}{(x-1)^4}\right\}y=0\text{.....(17)}.$$

This then is an equation two independent solutions of which are given by (5). The other two will be considered later. (See Art. 8.)

5. Let us now examine whether the last equation can be identical with (8). If (8) be written in a form in which the coefficient of $D^4y$ is unity, that of $D^3y$ is $-\frac{1}{Q}\frac{dQ}{dx}$; hence it is evident that the equations cannot be identical unless $Q$ be of the form $x^{-m}(x-1)^{-n}$, where $m$, $n$ are some constants, positive or negative; but the most general value of $P$ is of the form

$$P=\frac{a}{x^2}+\frac{b}{x(x-1)}+\frac{c}{(x-1)^2},$$

and of $Q$,

$$Q=\frac{a'}{x^2}+\frac{b'}{x(x-1)}+\frac{c'}{(x-1)^2};$$

hence there are six cases in which the above condition is satisfied, viz.

I. $m=1,\ n=2,\ a'=0,\ b'=-c',$ i.e. $\lambda^2=\lambda'^2,\ \mu^2=\mu'^2;$

II. $m=1,\ n=1,\ a'=0,\ c'=0,\qquad \lambda^2=\lambda'^2,\ \nu^2=\nu'^2;$

III. $m=2,\ n=1,\ c'=0,\ a'=-b',\qquad \mu^2=\mu'^2,\ \nu^2=\nu'^2;$

IV. $m=2,\ n=0,\ b'=0,\ c'=0,\qquad \nu^2=\nu'^2,\ \lambda^2-\mu^2=\lambda'^2-\mu'^2;$

V. $m=0,\ n=2,\ a'=0,\ b'=0,\qquad \lambda^2=\lambda'^2,\ \mu^2-\nu^2=\mu'^2-\nu'^2;$

VI. $m=2,\ n=2,\ c'=a',\ b'=-2a',\qquad \mu^2=\mu'^2,\ \lambda^2-\nu^2=\lambda'^2-\nu'^2.$

6. In Case I., writing $Q=c'x^{-1}(x-1)^{-2}$, equation (8) can be written in the form

$$D^4y+\left\{\frac{1}{x}+\frac{2}{x-1}\right\}D^3y+2\left\{\frac{a}{x^2}+\frac{b}{x(x-1)}+\frac{c}{(x-1)^2}\right\}D^2y$$
$$+\left\{\frac{-4a}{x^3}+\frac{4a-b}{x^2(x-1)}+\frac{b+2c}{x(x-1)^2}-\frac{2c}{(x-1)^3}\right\}Dy$$
$$+\left\{\frac{4a}{x^4}+\frac{b-4a}{x^3(x-1)}+\frac{-b+c'^2}{x^2(x-1)^2}-\frac{2c+2c'^2}{x(x-1)^3}+\frac{2c+c'^2}{(x-1)^4}\right\}y=0 \quad\text{.....(18).}$$

Comparing (17) and (18) we obtain 14 equations connecting the 16 quantities

$$P_1,\ Q_1,\ P_2,\ Q_2,\ R_2,\ P_3,\ Q_3,\ R_3,\ P_4,\ Q_4,\ R_4,\ c',\ a,\ b,\ c,\ \kappa.$$

Eliminating the 12 first we obtain the equations

$$(2\kappa+1)\{c+\kappa(\kappa+1)\}=0 \quad\text{.....(19),}$$

$$(2\kappa+1)\left\{(\kappa+1)(3\kappa-2)-b+2\frac{\kappa-1}{\kappa}c\right\}=0 \quad\text{.....(20),}$$

which are both satisfied by $\kappa = -\frac{1}{2}$,
and we thus obtain

$$\left.\begin{aligned}
P_1 &= 1 \\
Q_1 &= 4 \\
P_2 &= 2a \\
Q_2 &= 2b + \tfrac{3}{2} \\
R_2 &= 2c + \tfrac{3}{2} \\
P_3 &= -4a \\
Q_3 &= 6a - b \\
R_3 &= 3b + 2c - \tfrac{3}{4} \\
P_4 &= 4a \\
Q_4 &= b - 6a \\
R_4 &= \frac{3a - 3b - c}{2} + \frac{3}{16}
\end{aligned}\right\} \quad \text{.....(21)},$$

and

$$c'^2 = \frac{3}{16} - \frac{c}{2} \quad \text{.....(22)}.$$

The alternatives to $\kappa = -\frac{1}{2}$ involve two relations instead of one among $a$, $b$, $c$, $c'$, and thus cannot lead to cases of equal generality.

Using the above values in the 11 equations (15), connecting the 7 quantities $\theta'$, $\phi'$, $A'$, $B'$, $L$, $M$, $N$ with $a$, $b$, $c$, they are found to be satisfied for any values of $a$, $b$, $c$, by

$$\theta' = \tfrac{1}{2},\ \phi' = 1,\ A' = -1,\ B' = 0,\ L = 2a,\ M = 2b + 4a,\ N = 2a + 2b + 2c - \tfrac{3}{4} \quad \text{....(23)},$$

and cannot hold in any other case of equal generality.

The relations $$a' = 0,\ b' = -c',\ c'^2 = \frac{3}{16} - \frac{c}{2},$$

which characterize this case are equivalent to

$$\lambda = \pm\lambda',\quad \mu = \pm\mu',\quad \nu \pm \nu' = \pm 1 \quad \text{.....(24)}.$$

Consider first the case in which all these ambiguous signs are taken positive, that is to say, in which

$$\gamma = \gamma',\quad \alpha - \beta = \alpha' - \beta',\quad \alpha + \beta + \alpha' + \beta' = 2\gamma - 1 \quad \text{.....(25)}.$$

The values given in (23), when substituted in equations (14), lead to

$$\sigma = \pm\left(\frac{\nu - \nu'}{2}\right) = \pm(\nu - \tfrac{1}{2}) \quad \text{.....(26)}.$$

Here also we take the ambiguous sign positive and further obtain either

$$h = -\gamma \text{ or } h = \gamma - 2 \quad \text{.....(27)}.$$

Taking the former alternative we arrive at the results

$$\left.\begin{aligned} A &= 2\gamma - 1 \\ B &= 2\alpha + 2\beta - 1 \\ C &= 4\alpha\beta \\ \theta &= \gamma - \tfrac{1}{2} \\ \phi &= \gamma \end{aligned}\right\} \quad \text{.....(28).}$$

It will be found that if in case of the ambiguous equations (24), (26), (27) we make any other choice of alternatives than that above, the theorems thereby deducible will be of exactly the same type as those which follow.

If $z = F(\alpha'', \beta''; \gamma''; x)$ be a solution of equation (10) the values given in (28) are equivalent to the equations

$$\alpha'' = 2\alpha, \quad \beta'' = 2\beta, \quad \gamma'' = 2\gamma - 1 \quad \text{.....(29).}$$

7. We now insert the values we have found in the solution given by (5) of the equation (17) which has been proved identical with (8) whose solution is given by (9), and on dividing both sides by $(1-x)^{\frac{1}{2}} x^{\gamma}$ we obtain the theorem that if $z$ be *any* solution of the equation satisfied by $F(2\alpha, 2\beta; 2\gamma - 1; x)$ then $(\gamma - \frac{1}{2}; \gamma; z(1-x)^{\alpha+\beta+\frac{1}{2}-\gamma})$ is a linear function of the four independent functions which are the products of a solution of the equation satisfied by $F(\alpha, \beta; \gamma; x)$ and a solution of that satisfied by $F(\gamma - \frac{1}{2} - \alpha, \gamma - \frac{1}{2} - \beta; \gamma; x)$. If we denote the general solutions of the last-mentioned equations by $\eta$, $\eta'$, respectively, and use the suffixes 1, 2, &c. to distinguish the particular solutions as in Forsyth's *Differential Equations*, a consideration of the general forms of $z$, $\eta$, $\eta'$ shows that unless $2\gamma$ be an integer, positive or negative, if in this theorem we write $z = z_1 = z_2$, then the function of $\eta$, $\eta'$ involved is

$$\eta_1\eta_1' = \eta_1\eta_2' = \eta_2\eta_1' = \eta_2\eta_2',$$

and the theorem thus gives the equations

$$\{\gamma - \tfrac{1}{2}; \gamma; (1-x)^{\alpha+\beta+\frac{1}{2}-\gamma} F(2\alpha, 2\beta; 2\gamma - 1; x)\}$$

or

$$\begin{aligned} &\{\gamma - \tfrac{1}{2}; \gamma; (1-x)^{\gamma-\alpha-\beta-\frac{1}{2}} F(2\gamma - 2\alpha - 1, 2\gamma - 2\beta - 1; 2\gamma - 1; x)\} \\ &= F(\alpha, \beta; \gamma; x) F(\gamma - \alpha - \tfrac{1}{2}, \gamma - \beta - \tfrac{1}{2}; \gamma; x) \text{.....(30)} \\ &= (1-x)^{\alpha+\beta-\gamma+1} F(\alpha, \beta; \gamma; x) F(\alpha + \tfrac{1}{2}, \beta + \tfrac{1}{2}; \gamma; x) \text{.....(30')} \\ &= (1-x) F(\gamma - \alpha, \gamma - \beta; \gamma; x) F(\alpha + \tfrac{1}{2}, \beta + \tfrac{1}{2}; \gamma; x) \text{.....(30'')} \\ &= (1-x)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta; \gamma; x) F(\gamma - \alpha - \tfrac{1}{2}, \gamma - \beta - \tfrac{1}{2}; \gamma; x) \text{.....(30''').} \end{aligned}$$

The type of the theorem is the same whether the first or the second form of the left-hand member be used.

If we take the first form, Cayley's theorem and equation (30) are identical, and if in the latter $\gamma$ were changed into $\gamma + \frac{1}{2}$ they would be expressed by the same symbols.

If we write $z = z_3 = z_4$, then the function of $\eta$, $\eta'$ involved is

$$\eta_3\eta_3' = \eta_3\eta_4' = \eta_4\eta_3' = \eta_4\eta_4',$$

and the equations obtained are of exactly the same type as those above but expressed in different symbols.

It will also be found that we obtain equations of exactly the same type by choosing appropriate functions of $z$, $\eta$, $\eta'$ which proceed in descending powers of $x$.

8. A question naturally arises as to the other two solutions of (17) which are not given by (5). Its most general solution is of the form

$$y = (x-1)^{-\kappa} x^{-h} v,$$

where $v$ is given by (13). The relation (12) which exists among the coefficients when $v$ is expanded in powers, shows that there are values of $v$ which proceed in ascending powers of $x$ of which the first terms are respectively

$$1, \quad x^{1-A}, \quad x^{1-\phi}, \quad x^{2-\phi};$$

of these the first two are, but the last two cannot be, given by (5).

If however, reversing the train of substitutions by which $v$ is derived from $z$ we write $u' = (\phi;\ \theta;\ v)$ where $v$ is given by (13), the relation between successive coefficients in $u'$ is not (12) but the result of multiplying equation (12) by

$$(\theta + r)(\theta + r - 1)(\phi + r)(\phi + r - 1),$$

this relation being obtained from (13) in a manner similar to that in which (13) is obtained from (12).

The relation thus obtained leads to a differential equation for $u'$ identical not with (11) but with the result of performing on (11) the operation indicated by

$$(xD + \theta)(xD + \theta - 1)(xD + \phi)(xD + \phi - 1).$$

Such an equation is of the sixth order; four and only four of its solutions (combined linearly) are admissible from which to derive $v$ by means of the relation $v = (\theta;\ \phi;\ u)$ for our theorem. This equation is equivalent to that obtained by equating the left-hand member of (11) not to zero but to

$$C_1 x^{-\phi} + C_2 x^{1-\phi} + C_3 x^{-\theta} + C_4 x^{1-\theta} \quad\text{.....(31)},$$

$C_1$, $C_2$, $C_3$, $C_4$ being arbitrary constants. Changing in such an equation the value $u'$ into $z'$ by means of the relation $u' = (1-x)^{-\sigma} z'$ it becomes

$$[x(1-x)D^2 + (A - Bx)D - C]\, z' = (C_1 x^{-\phi} + C_2 x^{1-\phi} + C_3 x^{-\theta} + C_4 x^{1-\theta})(1-x)^{\sigma-1} \quad\text{...(32)}.$$

By expanding the right-hand member in powers of $x$ its solution can be deduced from that of a series of equations of the type

$$[x(1-x)D^2 + (A - Bx)D - C]\, z' = x^s,$$

or

$$[x(1-x)D^2 + (\gamma'' - (\alpha'' + \beta'' + 1)x)D - \alpha''\beta'']\, z' = x^s \quad\text{.....(33)}.$$

It may be easily established that a solution of this is

$$z' = \frac{x^{\epsilon+1}}{(\epsilon+\gamma'')(\epsilon+1)} + \frac{(\epsilon+\alpha''+1)(\epsilon+\beta''+1)}{(\epsilon+\gamma'')(\epsilon+\gamma''+1)(\epsilon+1)(\epsilon+2)} x^{\epsilon+2} + \dots,$$

which we will write in the form,

$$z' = \frac{x^{\epsilon+1}}{(\epsilon+\gamma'')(\epsilon+1)} F(\alpha''+\epsilon+1,\ \beta''+\epsilon+1;\ \gamma''+\epsilon+1,\ \epsilon+2;\ x) \dots\dots\dots\dots\dots(34).$$

It should be noted that the natural numbers do not here occur factorially in the denominators of the coefficients of the powers of $x$.

A solution of the equation

$$[x(1-x)D^2 + (\gamma'' - (\alpha''+\beta''+1)x)D - \alpha''\beta'']\, z' = x^{\epsilon}(1-x)^{\sigma-1} \dots\dots\dots\dots(35),$$

is therefore

$$\begin{aligned} z' = {} & \frac{x^{\epsilon+1}}{(\epsilon+\gamma'')(\epsilon+1)} F(\alpha''+\epsilon+1,\ \beta''+\epsilon+1;\ \gamma''+\epsilon+1,\ \epsilon+2;\ x) \\ & - \frac{(\sigma-1)x^{\epsilon+2}}{1.(\epsilon+\gamma''+1)(\epsilon+2)} F(\alpha''+\epsilon+2,\ \beta''+\epsilon+2;\ \gamma''+\epsilon+2,\ \epsilon+3;\ x) \\ & + \frac{(\sigma-1)(\sigma-2)x^{\epsilon+3}}{1.2.(\epsilon+\gamma''+2)(\epsilon+3)} F(\alpha''+\epsilon+3,\ \beta''+\epsilon+3;\ \gamma''+\epsilon+3,\ \epsilon+4;\ x) - \dots\dots(36), \end{aligned}$$

the complementary function being the most general value of $z$, as given by (10).

On forming $v$ by the relation

$$v = (\theta;\ \phi;\ (1-x)^{-\sigma} z') \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(37),$$

it appears that the leading term of $v$ has the same index as the leading term of $z'$ so that in order to obtain the expressions for $v$, the indices of whose leading terms are respectively $1-\phi$, $2-\phi$, we must (unless $\theta$ differs from $\phi$ by an integer which is not the case here) write $C_3 = C_4 = 0$.

The two values of $z'$ to be inserted in (37) in order to obtain thereby the missing solutions of (13) may thus be obtained from (36) by giving $\epsilon$ the values $-\phi$, $1-\phi$.

Since

$$(C_1 x^{-\phi} + C_2 x^{1-\phi})(1-x)^{\sigma-1} = (C_1 + C_2)x^{-\phi}(1-x)^{\sigma-1} - C_2 x^{-\phi}(1-x)^{\sigma},$$

another solution may be obtained from (36) by giving $\epsilon$ the value $-\phi$, and increasing $\sigma$ by unity.

9. The two linear relations which connect the two independent solutions of (17) last obtained with the other two solutions of (8) may of course be expressed in an infinite number of ways. We may specify the following

$$\begin{aligned} & (\gamma-1)\Big(\tfrac{1}{2};\ 1;\ (1-x)^{\alpha+\beta+\frac{1}{2}-\gamma}\Big\{\frac{1}{\gamma-1} F(2\alpha-\gamma+1,\ 2\beta-\gamma+1;\ \gamma,\ 2-\gamma;\ x) \\ & + \frac{\alpha+\beta+\frac{1}{2}-\gamma}{1} \frac{x}{\gamma} F(2\alpha-\gamma+2,\ 2\beta-\gamma+2;\ \gamma+1,\ 3-\gamma;\ x) \\ & + \frac{(\alpha+\beta+\frac{1}{2}-\gamma)(\alpha+\beta+\frac{3}{2}-\gamma)}{1.2} \frac{x^2}{\gamma+1} F(2\alpha-\gamma+3,\ 2\beta-\gamma+3;\ \gamma+2,\ 4-\gamma;\ x) + \dots\dots\Big\}\Big) \\ & \qquad = F(\alpha,\ \beta;\ \gamma;\ x).F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta;\ 2-\gamma;\ x) \dots\dots\dots\dots\dots\dots\dots\dots(38). \end{aligned}$$

The right-hand member of this equation and that of nearly all those which follow may be written in other forms, showing a power of $1-x$ as a factor.

By interchanging $\alpha$ and $\frac{1}{2}-\alpha$, $\beta$ and $\frac{1}{2}-\beta$, $\gamma$ and $2-\gamma$ we obtain another form for the left-hand member.

Another relation among the same solutions independent of the last may be obtained from it by interchanging $\alpha$ and $\gamma-\alpha-\frac{1}{2}$, $\beta$ and $\gamma-\beta-\frac{1}{2}$ .....(39).

10. If Case II. of Art. 5 be investigated in a similar manner it will be found that similar theorems hold provided in addition to the preliminary conditions $\lambda=\pm\lambda'$, $\nu=\pm\nu'$, the relation $\mu\pm\mu'=\pm 1$ also holds, and that without loss of generality we may as in Case I. take all these ambiguous signs positive. These conditions are equivalent to

$$\beta'=\alpha-\tfrac{1}{2},\quad \alpha'=\beta+\tfrac{1}{2},\quad \gamma'=\gamma \qquad\text{.....(40).}$$

In this case we obtain the following values:

$$\left.\begin{aligned}
&\sigma=\gamma-\alpha-\beta+\tfrac{1}{2}\ (\text{or } \sigma=\alpha+\beta-\gamma+\tfrac{1}{2}),\\
&h=-\gamma \qquad\qquad (\text{or } h=\gamma-2)\\
&\kappa=-1\\
&\theta=\gamma-\tfrac{1}{2}\\
&\phi=\gamma\\
&A=2\gamma-1\\
&B=2\alpha+2\beta\\
&C=2\beta(2\alpha-1),\\
&\text{the last three leading to}\quad \alpha''=2\alpha-1,\quad \beta''=2\beta,\quad \gamma''=2\gamma.
\end{aligned}\right\}\text{.....(41).}$$

As in Case I. the alternatives rejected only lead to theorems of the same type as do those chosen.

The equations which correspond to (30)—(30‴) in this case may be written in the form

$$\{\gamma-\tfrac{1}{2};\ \gamma;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}F(2\alpha-1,\ 2\beta;\ 2\gamma-1;\ x)\},$$

or

$$\begin{aligned}
&\{\gamma-\tfrac{1}{2};\ \gamma;\ (1-x)^{\gamma-\alpha-\beta-\frac{1}{2}}F(2\gamma-2\alpha,\ 2\gamma-2\beta-1;\ 2\gamma-1;\ x)\}\\
&=(1-x)^{\alpha+\beta-\gamma}F(\alpha,\beta;\ \gamma;\ x)\,.\,F(\alpha-\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma;\ x) \text{ .....(42)}\\
&=F(\alpha,\beta;\ \gamma;\ x)\,.\,F(\gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta-\tfrac{1}{2};\ \gamma;\ x) \text{ .....(42')}\\
&=F(\gamma-\alpha,\gamma-\beta;\ \gamma;\ x)\,.\,F(\alpha-\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma;\ x) \text{ .....(42'')}\\
&=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\ \gamma;\ x)\,.\,F(\gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta-\tfrac{1}{2};\ \gamma;\ x) \text{ .....(42''').}
\end{aligned}$$

As in Case I. if we consider the product of those solutions of the equations satisfied by $F(\alpha,\beta;\gamma;x)$ and $F(\alpha-\frac{1}{2},\beta+\frac{1}{2};\gamma;x)$ which, proceeding in ascending powers of $x$, begin with $x^{1-\gamma}$, the equations obtained are of exactly the same type as these, but expressed in different symbols.

Relations among the other two solutions of (8) and of (17) may, after dividing by $x^{1-\gamma}$, be written in the form

$$\begin{aligned}
&(1+2\beta-2\alpha)(\gamma-1)\Big\{\tfrac{1}{2};\ 1;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\Big(\frac{1}{\gamma-1}F(2\alpha-\gamma,\ 2\beta-\gamma+1;\ \gamma,\ 2-\gamma;\ x)\\
&\quad+\frac{\alpha+\beta-\gamma-\frac{1}{2}}{1}\frac{x}{\gamma}F(2\alpha-\gamma+1,\ 2\beta-\gamma+2;\ \gamma+1,\ 3-\gamma;\ x)\\
&\quad+\frac{(\alpha+\beta-\gamma-\frac{1}{2})(\alpha+\beta-\gamma+\frac{1}{2})}{1\,.\,2}\frac{x^2}{\gamma+1}F(2\alpha-\gamma+2,\ 2\beta-\gamma+3;\ \gamma+2,\ 4-\gamma;\ x)+\dots\Big)\Big\}\\
&=(2-2\alpha)F(\alpha,\beta;\ \gamma;\ x)\,.\,F(\tfrac{3}{2}-\alpha,\ \tfrac{1}{2}-\beta;\ 2-\gamma;\ x)\\
&\qquad+(2\beta-1)F(1-\alpha,\ 1-\beta;\ 2-\gamma;\ x)\,.\,F(\alpha-\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma;\ x)\text{.....(43).}
\end{aligned}$$

Another independent relation may be obtained by writing in the above

$$\alpha-\gamma+1 \text{ for } \alpha, \quad \beta-\gamma+1 \text{ for } \beta, \quad 2-\gamma \text{ for } \gamma,$$

and using an obvious alternative form of the right-hand member .....(44).

Another relation among the same solutions may be deduced from (43) by changing

$$\alpha \text{ into } \gamma-\alpha+\tfrac{1}{2} \text{ and } \beta \text{ into } \gamma-\beta-\tfrac{1}{2},$$

and using a similar alternative form of the right-hand member.....(45).

A fourth relation may be obtained from (43) by changing

$$\alpha \text{ into } \tfrac{3}{2}-\alpha, \quad \beta \text{ into } \tfrac{1}{2}-\beta, \quad \gamma \text{ into } 2-\gamma \text{.....(46).}$$

The left-hand members of (45), (46) involve the alternative value of $\sigma$ [see (41) and (36)] and the corresponding values of $\alpha''$, $\beta''$, $\gamma''$.

By taking the solutions of equations (8) and (17) which proceed in ascending powers commencing with $x^{2-\gamma}$ we obtain a fifth relation which, after dividing by $x^{1-\gamma}$, may be written in the form

$$\begin{aligned}
x\Big\{\tfrac{3}{2};\ 2;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\Big(&\frac{1}{(2-\gamma)\gamma}F(2\alpha-\gamma+1,\ 2\beta-\gamma+2;\ \gamma+1,\ 3-\gamma;\ x)\\
&+\frac{\alpha+\beta-\gamma+\frac{1}{2}}{(3-\gamma)(\gamma+1)1}xF(2\alpha-\gamma+2,\ 2\beta-\gamma+3;\ \gamma+2,\ 4-\gamma;\ x)\\
&+\frac{(\alpha+\beta-\gamma+\frac{1}{2})(\alpha+\beta-\gamma+\frac{3}{2})}{(4-\gamma)(\gamma+2)1.2}x^2F(2\alpha-\gamma+3,\ 2\beta-\gamma+4;\ \gamma+3,\ 5-\gamma;\ x)+\ldots\Big)\Big\}\\
=\frac{2}{(1+2\beta-2\alpha)(1-\gamma)}&[F(\alpha,\ \beta;\ \gamma;\ x).F(\tfrac{3}{2}-\alpha,\ \tfrac{1}{2}-\beta;\ 2-\gamma;\ x)\\
&-F(\alpha-\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma;\ x).F(1-\alpha,\ 1-\beta;\ 2-\gamma;\ x)]\text{.....(47).}
\end{aligned}$$

As the right-hand member is unaltered by changing at the same time $\alpha$ into $1-\beta$, $\beta$ into $1-\alpha$, and $\gamma$ into $2-\gamma$, so also must the left.

11. If in Case II. we choose appropriate solutions proceeding in descending powers of $x$ we arrive at Case III., in which similar theorems hold provided in addition to the preliminary conditions $\mu=\pm\mu'$, $\nu=\pm\nu'$, the further relation $\lambda\pm\lambda'=\pm 1$ is satisfied.

This case may also be investigated independently in the same manner as the others.

The equations which in this case correspond to (30)—(30‴) may be written in the form

$$\{\gamma+\tfrac{1}{2};\ \gamma+1;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}F(2\alpha,\ 2\beta;\ 2\gamma;\ x)\},$$

or

$$\begin{aligned}
&\{\gamma+\tfrac{1}{2};\ \gamma+1;\ (1-x)^{\gamma-\alpha-\beta-\frac{1}{2}}F(2\gamma-2\alpha,\ 2\gamma-2\beta;\ 2\gamma;\ x)\}\\
&=F(\gamma-\alpha,\ \gamma-\beta;\ \gamma;\ x).F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma+1;\ x)\text{.....(48)}\\
&=(1-x)^{\alpha+\beta-\gamma}F(\alpha,\ \beta;\ \gamma;\ x).F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma+1;\ x)\text{.....(48')}\\
&=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta;\ \gamma;\ x).F(\gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta+\tfrac{1}{2};\ \gamma+1;\ x)\ \ldots(48'')\\
&=F(\alpha,\ \beta;\ \gamma;\ x).F(\gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta+\tfrac{1}{2};\ \gamma+1;\ x)\text{.....(48‴).}
\end{aligned}$$

A relation among the other two solutions of (8) and of (17) may, on division by $x^{-\gamma}$, be written in the form

$$(\gamma-1)\left\{\tfrac{1}{2};\ 1;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\left(\frac{1}{\gamma-1}F(2\alpha-\gamma,\ 2\beta-\gamma;\ \gamma,\ 1-\gamma;\ x)\right.\right.$$

$$+\frac{\alpha+\beta-\gamma-\frac{1}{2}}{1}\frac{x}{\gamma}F(2\alpha-\gamma+1,\ 2\beta-\gamma+1;\ \gamma+1,\ 2-\gamma;\ x)$$

$$\left.\left.+\frac{(\alpha+\beta-\gamma-\frac{1}{2})(\alpha+\beta-\gamma+\frac{1}{2})}{1.2}\frac{x^2}{\gamma+1}F(2\alpha-\gamma+2,\ 2\beta-\gamma+2;\ \gamma+2,\ 3-\gamma;\ x)+\ldots\right)\right\}$$

$$=F(\alpha,\ \beta;\ \gamma;\ x).F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta;\ 1-\gamma;\ x)$$

$$+\frac{(\alpha-\frac{1}{2})(\beta-\frac{1}{2})}{\gamma(1-\gamma)}xF(1-\alpha,\ 1-\beta;\ 2-\gamma;\ x).F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma+1;\ x)\ldots(49).$$

Another relation among the same solutions may be obtained from this by changing

$\alpha$ into $\frac{1}{2}-\alpha$, $\beta$ into $\frac{1}{2}-\beta$, $\gamma$ into $1-\gamma$ .....(50).

A third relation among the same solutions may be obtained from (49) by changing

$\alpha$ into $\gamma-\alpha$, $\beta$ into $\gamma-\beta$,

and using an obvious alternative form of the right-hand member .....(51).

A fourth relation may be obtained from (49) by changing

$\alpha$ into $\alpha+\frac{1}{2}-\gamma$, $\beta$ into $\beta+\frac{1}{2}-\gamma$, $\gamma$ into $1-\gamma$,

and using an obvious alternative form of the right-hand member .....(52).

A fifth relation on division by $x^{1-\gamma}$ may be written in the form

$$\gamma(1-\gamma)\left\{\tfrac{3}{2};\ 2;\ (1-x)^{\alpha+\beta-\gamma-\frac{1}{2}}\left(\frac{1}{\gamma(1-\gamma)}F(2\alpha-\gamma+1,\ 2\beta-\gamma+1;\ \gamma+1,\ 2-\gamma;\ x)\right.\right.$$

$$+\frac{\alpha+\beta-\gamma+\frac{1}{2}}{1}\frac{x}{(\gamma+1)(2-\gamma)}F(2\alpha-\gamma+2,\ 2\beta-\gamma+2;\ \gamma+2,\ 3-\gamma;\ x)$$

$$\left.\left.+\frac{(\alpha+\beta-\gamma+\frac{1}{2})(\alpha+\beta-\gamma+\frac{3}{2})}{1.2}\frac{x^2}{(\gamma+2)(3-\gamma)}F(2\alpha-\gamma+3,\ 2\beta-\gamma+3;\ \gamma+3,\ 4-\gamma;\ x)+\ldots\right)\right\}$$

$$=F(1-\alpha,\ 1-\beta;\ 2-\gamma;\ x).F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \gamma+1;\ x)\ldots\ldots\ldots\ldots(53)$$

$$=F(\alpha-\gamma+1,\ \beta-\gamma+1;\ 2-\gamma;\ x).F(\gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta+\tfrac{1}{2};\ \gamma+1;\ x)\ \ldots\ldots(53').$$

By interchanging $\alpha$ and $\gamma-\alpha$, $\beta$ and $\gamma-\beta$, which does not alter the value of the right-hand member, we obtain another form of the left-hand member.

12. Of the three relations which have been obtained in Case I. in equations (30)—(30‴), in Case II. in equations (42)—(42‴), in Case III. in equations (48)—(48‴), any two can be deduced from the remaining one by changing the constants and using the relations connecting three hypergeometric series whose constants differ by integers (Gauss, *Complete Works*, Vol. III. p. 133).

13. Cases IV., V., VI. of Art. 5, will be found on examination to lead to no theorems of equal generality with those above.

14. We next proceed to consider some special cases in which the relations we have obtained can be expressed more simply.

Suppose in Case I. we have $\gamma = \alpha + \beta + \frac{1}{2}$. Using this relation in equations (30)—(30‴) the left-hand member becomes a hypergeometric series of the third order and we obtain the relations

$$F(2\alpha, 2\beta, \alpha+\beta;\ 2\alpha+2\beta, \alpha+\beta+\tfrac{1}{2};\ x) = [F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x)]^2 \quad \text{.....(54)}$$

$$= (1-x)^{\frac{1}{2}} F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) \,.\, F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \quad \text{.....(54')}$$

$$= (1-x)\,[F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x)]^2 \quad \text{.....(54'').}$$

The natural numbers *do* occur factorially in the denominators in this hypergeometric series of the third order and in others below.

Equation (38) now identical with (39) becomes

$$F(\alpha-\beta+\tfrac{1}{2}, \beta-\alpha+\tfrac{1}{2}, \tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2}, \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$= F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) \,.\, F(\tfrac{1}{2}-\alpha, \tfrac{1}{2}-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x) \quad \text{.....(55),}$$

which can also be expressed in other ways by taking out a power of $1-x$ as a factor.

It may be shown more easily directly that the linear differential equation satisfied by the square of the hypergeometric series $F(\alpha, \beta;\ \gamma;\ x)$ becomes identical with that of a hypergeometric series of the third order if, and only if, $\gamma = \alpha+\beta+\frac{1}{2}$. This has been done by von Clausen, *Crelle's Journal*, Vol. III., where equations (54), (55) are given. On equating the appropriate third independent solutions of the two equations to each other the result is of the same type as (54)—(54'') expressed in different symbols.

If $\gamma = \alpha+\beta-\frac{1}{2}$ relations are obtained of the same type as (54)—(55) expressed in different symbols.

If in Case II. the same relation $\gamma = \alpha+\beta+\frac{1}{2}$ holds, equations (42)—(42‴) become, taking the second form of the left-hand member,

$$F(2\alpha, 2\beta+1, \alpha+\beta;\ 2\alpha+2\beta, \alpha+\beta+\tfrac{1}{2};\ x)$$
$$= F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) \,.\, F(\alpha, \beta+1;\ \alpha+\beta+\tfrac{1}{2};\ x) \quad \text{.....(56)}$$
$$= F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \,.\, F(\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \quad \text{.....(56').}$$

Equations (45), (46) are now identical and reduce to the form

$$(1+2\beta-2\alpha)\, F(\beta-\alpha+\tfrac{3}{2}, \alpha-\beta+\tfrac{1}{2}, \tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2}, \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$= (1-2\alpha)\, F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) \,.\, F(\tfrac{3}{2}-\alpha, \tfrac{1}{2}-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$+ 2\beta F(\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \,.\, F(1-\alpha, 1-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x) \quad \text{.....(57),}$$

in which the right-hand member may be written in a variety of forms.

Equation (47) may now be written in the form

$$F(\alpha-\beta-\tfrac{1}{2}, \beta-\alpha+\tfrac{1}{2}, \tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2}, \tfrac{3}{2}-\alpha-\beta;\ x) - 1$$
$$= \frac{(1+2\beta-2\alpha)}{2(2\alpha+2\beta-1)} [F(\alpha, \beta;\ \alpha+\beta+\tfrac{1}{2};\ x) \,.\, F(\tfrac{3}{2}-\alpha, \tfrac{1}{2}-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$- F(\alpha-\tfrac{1}{2}, \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x) \,.\, F(1-\alpha, 1-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x)] \quad \text{.....(58).}$$

If instead of the relation $\gamma=\alpha+\beta+\frac{1}{2}$, $\gamma=\alpha+\beta-\frac{1}{2}$ holds, equations (43), (44) reduce to one of the same type as (57) expressed in different symbols, while (47) again reduces to one of the same type as (58).

In Case III. also if we suppose $\gamma=\alpha+\beta+\frac{1}{2}$ or $\gamma=\alpha+\beta-\frac{1}{2}$, the results already obtained may be expressed more simply and both suppositions lead to equations of the same type; we choose the latter. Equations (48)—(48‴) then become

$$F(2\alpha,\ 2\beta,\ \alpha+\beta;\ 2\alpha+2\beta-1,\ \alpha+\beta+\tfrac{1}{2};\ x)$$
$$=F(\alpha-\tfrac{1}{2},\ \beta-\tfrac{1}{2};\ \alpha+\beta-\tfrac{1}{2};\ x)\,.\,F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x)\ldots\ldots\ldots\ldots(59)$$
$$=F(\alpha,\ \beta;\ \alpha+\beta-\tfrac{1}{2};\ x)\,.\,F(\alpha,\ \beta;\ \alpha+\beta+\tfrac{1}{2};\ x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(59').$$

Equations (49) and (52) are now identical and reduce to

$$F(\alpha-\beta+\tfrac{1}{2},\ \beta-\alpha+\tfrac{1}{2},\ \tfrac{1}{2};\ \alpha+\beta-\tfrac{1}{2},\ \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$=F(\alpha,\ \beta;\ \alpha+\beta-\tfrac{1}{2};\ x)\,.\,F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta;\ \tfrac{3}{2}-\alpha-\beta;\ x)$$
$$+\frac{(\alpha-\frac{1}{2})(\beta-\frac{1}{2})}{(\alpha+\beta-\frac{1}{2})(\frac{3}{2}-\alpha-\beta)}\,xF(1-\alpha,\ 1-\beta;\ \tfrac{5}{2}-\alpha-\beta;\ x)\,.\,F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x)\ldots(60).$$

Equations (53), (53′) now become of the form

$$\{\tfrac{3}{2};\ 2;\ (1-x)^{-1}F(\alpha-\beta+\tfrac{1}{2},\ \beta-\alpha+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2},\ \tfrac{5}{2}-\alpha-\beta;\ x)\}$$
$$=F(1-\alpha,\ 1-\beta;\ \tfrac{5}{2}-\alpha-\beta;\ x)\,.\,F(\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \alpha+\beta+\tfrac{1}{2};\ x)\ldots\ldots\ldots\ldots(61)$$
$$=F(\tfrac{3}{2}-\alpha,\ \tfrac{3}{2}-\beta;\ \tfrac{5}{2}-\alpha-\beta;\ x)\,.\,F(\alpha,\ \beta;\ \alpha+\beta+\tfrac{1}{2};\ x)\ \ldots\ldots\ldots\ldots\ldots\ldots(61').$$

The relations obtained also admit of some simplification in all three cases if $\gamma$ and $\alpha+\beta$ differ by half any odd integer.

15. We have seen in the preceding article that if $\gamma=\alpha+\beta+\frac{1}{2}$ all the solutions of the linear differential equation satisfied by $[F(\alpha,\ \beta;\ \gamma;\ x)]^2$ are solutions of an equation of the type of (17); and, which is the same thing, that if $\gamma=\alpha+\beta-\frac{1}{2}$ all the solutions of the differential equation satisfied by $(1-x)[F(\alpha,\ \beta;\ \gamma;\ x)]^2$ are solutions of an equation of the same type; the theorems deducible in these cases being stated in (54)—(55).

The cases discussed include two others in which all the solutions of the equation satisfied by $(1-x)^\alpha F[\alpha,\ \beta;\ \gamma;\ x]^2$ are solutions of (17), viz. if in Case II. we write $\beta+\frac{1}{2}=\alpha$, and if in Case III. we write $\gamma=\frac{1}{2}$ (or $\frac{3}{2}$); it seems unnecessary to give the forms which equations (42)—(53′) then assume.

The theorems obtained involving the square of a hypergeometric series require that either $\lambda$, $\mu$, or $\nu$ should be $\pm\frac{1}{2}$.

It appears probable that in other cases than these three all the solutions of the equation satisfied by $(1-x)^\alpha[F(\alpha,\ \beta;\ \gamma;\ x)]^2$ satisfy an equation like (17).

16. In connection with the preceding remark the writer has investigated in what cases the linear equation satisfied by $[F(\alpha,\ \beta;\ \gamma;\ x)]^2$ may have one or two solutions but not all three in the form of a hypergeometric series of the third order. The method pursued, (that of examining in what cases the successive coefficients of the

latter series can obey the law required in $F^2$), was very tedious; it is believed however that the following cases include all.

The existing relations in the various cases, after division in some instances by a power of $x$, may be written as follows:

Case I. $\nu=\gamma-\alpha-\beta=\frac{3}{2}$, $\lambda=1-\gamma=\pm\frac{1}{2}$.

$$\left[F\left(\frac{-1+\mu}{2}, \frac{-1-\mu}{2}; \tfrac{1}{2}; x\right)\right]^2-\left(\frac{1-\mu^2}{\mu}\right)^2 x\left[F\left(\frac{\mu}{2}, -\frac{\mu}{2}; \tfrac{3}{2}; x\right)\right]^2$$

$$=F(\mu-1, -\mu-1, 1-2\mu^2; \tfrac{1}{2}, -2\mu^2; x) \quad\text{.....(62),}$$

or

$$=F(\mu-1, -\mu-1; \tfrac{1}{2}; x)+\frac{\mu^2-1}{\mu^2}xF(\mu, -\mu; \tfrac{3}{2}; x)\text{.....(62'),}$$

$$F\left(\frac{-1+\mu}{2}, \frac{-1-\mu}{2}; \tfrac{1}{2}; x\right).F\left(\frac{\mu}{2}, -\frac{\mu}{2}; \tfrac{3}{2}; x\right)=F(\mu-\tfrac{1}{2}, -\mu-\tfrac{1}{2}, \tfrac{3}{2}-2\mu^2; \tfrac{3}{2}, \tfrac{1}{2}-2\mu^2; x)\text{...(63)}$$

$$=F(\mu-\tfrac{1}{2}, -\mu-\tfrac{1}{2}; \tfrac{3}{2}; x)+\tfrac{1}{3}xF(\mu+\tfrac{1}{2}, -\mu+\tfrac{1}{2}; \tfrac{5}{2}; x)\text{.....(63'),}$$

$$\left[F\left(\frac{-1+\mu}{2}, \frac{-1-\mu}{2}; \tfrac{1}{2}; x\right)\right]^2+\left(\frac{1-\mu^2}{\mu}\right)^2 x\left[F\left(\frac{\mu}{2}, -\frac{\mu}{2}; \tfrac{3}{2}; x\right)\right]^2$$

$$=F(-1, \mu^2-1; \mu^2; x)=1+\frac{1-\mu^2}{\mu^2}x \quad\text{.....(64).}$$

Case II. $\nu=\gamma-\alpha-\beta=\frac{3}{2}$, $\mu=\alpha-\beta=\pm\frac{1}{2}$.

$$\left[F\left(-\frac{\lambda}{2}, \frac{-\lambda-1}{2}; 1-\lambda; x\right)\right]^2=F(-\lambda-1, -\lambda-\tfrac{1}{2}, 2\lambda^2-\lambda; -2\lambda+1, 2\lambda^2-\lambda-1; x)\text{...(65)}$$

$$=F(-\lambda-1, -\lambda-\tfrac{1}{2}; -2\lambda+1; x)-\frac{\lambda+1}{2(1-\lambda)(1-2\lambda)}xF(-\lambda, -\lambda+\tfrac{1}{2}; -2\lambda+2; x)\text{...(65').}$$

Another relation of the same type obtainable by changing the sign of $\lambda$.....(66).

And

$$F\left(-\frac{\lambda}{2}, \frac{-\lambda-1}{2}; 1-\lambda; x\right).F\left(\frac{\lambda}{2}, \frac{\lambda-1}{2}; 1+\lambda; x\right)=1+\frac{\lambda^2}{1-\lambda^2}x\text{.....(67).}$$

Case III. $\mu=\frac{3}{2}$, $\lambda=\pm\frac{1}{2}$.

$$F\left(-\frac{\nu}{2}, \frac{-\nu+3}{2}; \tfrac{3}{2}; x\right).F\left(\frac{-\nu-1}{2}, \frac{-\nu+2}{2}; \tfrac{1}{2}; x\right)$$

$$=F\left(-\nu-\tfrac{1}{2}, -\nu+1, \frac{-2\nu^2+3\nu+3}{2\nu+2}; \tfrac{3}{2}, \frac{-2\nu^2+\nu+1}{2\nu+2}; x\right) \quad\text{.....(68)}$$

$$=F(-\nu-\tfrac{1}{2}, -\nu+1; \tfrac{3}{2}; x)-\frac{2\nu+2}{3}xF(-\nu+\tfrac{1}{2}, -\nu+2; \tfrac{5}{2}; x)\text{.....(68'),}$$

$$\left[F\left(\frac{-\nu-1}{2}, \frac{-\nu+2}{2}; \tfrac{1}{2}; x\right)\right]^2+\left(\frac{\nu^2-1}{\nu}\right)^2 x\left[F\left(-\frac{\nu}{2}, \frac{-\nu+3}{2}; \tfrac{3}{2}; x\right)\right]^2$$

$$=F\left(-\nu-1, -\nu+\tfrac{1}{2}, \frac{-\nu^2+\nu+1}{\nu+1}; \tfrac{1}{2}, -\frac{\nu^2}{\nu+1}; x\right) \quad\text{.....(69)}$$

$$=F(-\nu-1, -\nu+\tfrac{1}{2}; \tfrac{1}{2}; x)+\frac{(\nu+1)^2(-2\nu+1)}{\nu^2}xF(-\nu, -\nu+\tfrac{3}{2}; \tfrac{3}{2}; x)\text{...(69'),}$$

$$\left[F\left(\frac{-\nu-1}{2}, \frac{-\nu+2}{2}; \tfrac{1}{2}; x\right)\right]^2 - \left(\frac{\nu^2-1}{\nu}\right)^2 x \left[F\left(-\frac{\nu}{2}, \frac{-\nu+3}{2}; \tfrac{3}{2}; x\right)\right]^2$$
$$= F\left(-\nu-1, \frac{\nu^2-\nu+1}{-\nu+1}; \frac{\nu^2}{-\nu+1}; x\right) = (1-x)^\nu \left(1-\frac{x}{\nu^2}\right) \quad \text{.....(70).}$$

Case IV. $\mu = \frac{1}{2}$, $\lambda = \pm\frac{3}{2}$.

$$F\left(-\frac{\nu}{2}, \frac{-\nu-1}{2}; -\tfrac{1}{2}; x\right).F\left(\frac{2-\nu}{2}, \frac{3-\nu}{2}; \tfrac{5}{2}; x\right) = F\left(\tfrac{1}{2}-\nu, 1-\nu, \frac{\nu+3}{2\nu+2}; \tfrac{5}{2}, \frac{1-\nu}{2\nu+2}; x\right) \text{...(71)}$$
$$= F(\tfrac{1}{2}-\nu, 1-\nu; \tfrac{5}{2}; x) + \frac{2(1+\nu)(1-2\nu)}{5} x F(\tfrac{3}{2}-\nu, 2-\nu; \tfrac{7}{2}; x) \text{.....(71')},$$

$$\left[F\left(-\frac{\nu}{2}, \frac{-\nu-1}{2}; -\tfrac{1}{2}; x\right)\right]^2 + \left(\frac{\nu(\nu^2-1)}{3}\right)^2 x^2 \left[F\left(\frac{2-\nu}{2}, \frac{3-\nu}{2}; \tfrac{5}{2}; x\right)\right]^2$$
$$= F\left(-\nu-1, -\nu-\tfrac{1}{2}, -\frac{\nu}{\nu+1}; -\tfrac{1}{2}, \frac{-2\nu-1}{\nu+1}; x\right) \text{.....(72)}$$
$$= F(-\nu-1, -\nu-\tfrac{1}{2}; -\tfrac{1}{2}; x) + (\nu+1)^2 x F(-\nu, \tfrac{1}{2}-\nu; \tfrac{1}{2}; x) \text{.....(72')},$$

$$\left[F\left(-\frac{\nu}{2}, \frac{-\nu-1}{2}; -\tfrac{1}{2}; x\right)\right]^2 - \left(\frac{\nu(\nu^2-1)}{3}\right)^2 x^2 \left[F\left(\frac{2-\nu}{2}, \frac{3-\nu}{2}; \tfrac{5}{2}; x\right)\right]^2$$
$$= F\left(-\nu-1, -\frac{\nu}{1-\nu}; -\frac{1}{1-\nu}; x\right) = (1-x)^\nu (1-\nu^2 x) \quad \text{.....(73).}$$

Case V. $\mu = \frac{1}{2}$, $\lambda = \pm\frac{1}{2}$.

$$F\left(-\frac{\nu}{2}, \frac{1-\nu}{2}; \tfrac{1}{2}; x\right).F\left(\frac{1-\nu}{2}, \frac{2-\nu}{2}; \tfrac{3}{2}; x\right) = F(\tfrac{1}{2}-\nu, 1-\nu; \tfrac{3}{2}; x) \text{.....(74)},$$

$$\left[F\left(-\frac{\nu}{2}, \frac{1-\nu}{2}; \tfrac{1}{2}; x\right)\right]^2 + \nu^2 x \left[F\left(\frac{1-\nu}{2}, \frac{2-\nu}{2}; \tfrac{3}{2}; x\right)\right]^2 = F(-\nu, \tfrac{1}{2}-\nu; \tfrac{1}{2}; x) \text{...(75)},$$

$$\left[F\left(-\frac{\nu}{2}, \frac{1-\nu}{2}; \tfrac{1}{2}; x\right)\right]^2 - \nu^2 x \left[F\left(\frac{1-\nu}{2}, \frac{2-\nu}{2}; \tfrac{3}{2}; x\right)\right]^2 = (1-x)^\nu \quad \text{.....(76).}$$

In each of the foregoing five cases the equation satisfied by $[F(\alpha, \beta; \gamma; x)]^2$ has two independent solutions which are also solutions of that satisfied by a hypergeometric series of the third order and a third independent solution of the type $x^a(1-x)^b(1+cx)$.

If in Cases I., II. the left-hand member be transformed by means of the identity $F(\alpha, \beta; \gamma; x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; x)$, we obtain Case VI. $\nu = -\frac{3}{2}$, $\lambda = \pm\frac{1}{2}$, and Case VII. $\nu = -\frac{3}{2}$, $\mu = \pm\frac{1}{2}$. In each of these last cases the differential equation for the square of a hypergeometric series has one solution which is a solution of the equation satisfied by another hypergeometric series of the second order and is in fact of the form $x^a(1-x)^b(1+cx)$; two other independent solutions in these cases are the product of a power of $1-x$ and a solution of another hypergeometric series of the third order. It does not seem necessary to give the results in these cases.

# II. *On the possibility of deducing magneto-optic phenomena from a direct modification of an electro-dynamic energy function.* By J. G. LEATHEM, M.A., Fellow of St John's College.

[*Received* and *read* 16 May 1898.]

## INTRODUCTION.

1. THE method initiated by Maxwell for the explanation of the Faraday effect depended on the direct insertion of a magneto-optic term in the energy. This method was extended by FitzGerald[1] and others to the explanation of Kerr's effect, namely the modification introduced in the circumstances of optical reflexion by magnetisation of the reflector. A difficulty occurred however in satisfying all the interfacial conditions, which virtually shewed that such a scheme was not formally self-consistent. The origin of the discrepancy has been traced by Mr. Larmor[2] to omission to secure what may for shortness be called the electromotive incompressibility of the medium: in the ordinary problem of optical reflexion there is no tendency for this to be disturbed, but when Maxwell's magneto-optic energy terms are included the reaction against compression introduces what may be termed an electric pressure, which must appear in the equations. It was necessary to compare the modified scheme thus obtained with experimental knowledge: and the calculations given in this paper shew that in fact it does not represent the phenomena.

The paper is only a summary of the actual calculations; because since they were completed I have shewn[3] that the other rigorous theory formulated as an alternative by Mr. Larmor[4], which leads to a system of equations practically the same as those advanced on various hypotheses by FitzGerald, Goldhammer, Basset, Drude, and others, is in much more satisfactory agreement with experiment.

[1] G. F. FitzGerald, *Phil. Trans.* 1880.

[2] "Report on the Action of Magnetism on Light," *Brit. Assoc. Rep.* 1893.

[3] "On the magneto-optic phenomena of Iron, Nickel, and Cobalt," *Phil. Trans.* 1897. Considerable discussion has taken place in Wiedemann's *Annalen*, both before and subsequent to the publication of this memoir, on the question of the formal identity of the schemes of equations employed by these various writers. That this should have been possible is in itself a sufficient indication of the obscurity in which the fundamental principles of the subject had become enveloped. I had myself stated in the memoir that my system of equations was, as far as I could judge, formally the same as those of Goldhammer and Drude: but the theoretical principles from which they were derived, though bearing considerable resemblance to those of the former of these writers, seemed to me to be free from the empirical and tentative character of both. Indeed the relations of these theories to each other and to the one which I adopted had been fully indicated by Mr Larmor in 1893 in the Report above referred to.

[4] *Loc. cit.*

This brief history of the subject shews the desirability of also examining how far the former method of explanation agrees with the phenomena. The result is however what was to be expected by those who adhere to the more recent formulation of optical theory[1], which treats a material medium as free aether pervaded by discrete molecules involving in their constitution electrons considered as nuclei of intrinsic aethereal strain: on such a view a continuous energy function is not the starting point, and the influence of these discrete nuclei could hardly be expected to modify the propagation in the intervening aether in so fundamental a manner as an electromotive pressure would demand.

2. Mr. Larmor's modification of FitzGerald's scheme consists in the introduction of a new quantity $\lambda$, naturally suggested by the analysis, which may be interpreted as an irrotational or pressural wave propagated along the surface of separation of two different media. In the Report above referred to he obtains on this hypothesis the equations of propagation in a dielectric, and the conditions which must be satisfied at an interface between two non-conducting media. It is unnecessary to recapitulate here the results arrived at; but it may be well to mention that while in §§ 8 and 11, Mr. Larmor inadvertently states that $\lambda$ must be continuous across a bounding surface, he has since pointed out that this is not the case, as the pressure is not $\lambda$ but is the coefficient of $\delta\zeta$ in the variation of the action integral, and it is this pressure which must be continuous.

In the present paper it is proposed, by using the principle of Least Action and introducing a Dissipation Function, to obtain from the above hypotheses the differential equations of propagation in a *conducting* medium, and the boundary conditions which hold good at an interface between two such media. These will then be applied to the solution of the problem of the reflexion of light from a magnet; and the formulae so arrived at will be compared with the available experiments in this subject, with a view to testing to what extent the theory is capable of accounting for the observed phenomena.

## NOTATION, AND ASSUMPTIONS.

3. The notation is the same as Maxwell's; and there are introduced quantities $\xi$, $\eta$, $\zeta$ defined by the relation

$$(\alpha, \beta, \gamma) = d/dt\,(\xi, \eta, \zeta).$$

For brevity we put

$$\alpha_0 \frac{d}{dx} + \beta_0 \frac{d}{dy} + \gamma_0 \frac{d}{dz} \equiv \frac{d}{d\theta},$$

$(\alpha_0, \beta_0, \gamma_0)$ being the intensity of the imposed magnetisation; this is slightly different from the definition of $d/d\theta$ given by FitzGerald, but the alteration is justified by the consideration that magneto-optic rotations are proportional, not to the magnetic force, but to the intensity of magnetisation of the material medium.

[1] Cf. Larmor, "A Dynamical Theory of the Electric and Luminiferous Medium," Part III. *Phil. Trans.* 1898.

The form of $T'$, the magneto-optic part of the energy, given by FitzGerald (Larmor's Report, § 9) applies only to insulating media: it will be assumed that the corresponding expression for conducting media is

$$T' = C\iiint\left\{\frac{d\xi}{d\theta}\frac{d}{dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+\frac{d\eta}{d\theta}\frac{d}{dt}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+\frac{d\zeta}{d\theta}\frac{d}{dt}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}dx\,dy\,dz.$$

This is one of several possible forms of $T'$ which, for the particular case of a non-conducting medium, would be identical with that assumed by FitzGerald; it is of no importance which we choose, as they differ from one another only as to a complex factor in the magneto-optic constant $C$.

## Equations of Propagation, and Boundary Conditions.

4. The equations of propagation and boundary conditions are to be derived by the principle of Least Action from the energy functions $T$, $T'$, and $W$; but for conducting media it is necessary to combine with these Rayleigh's Dissipation Function.

The Dissipation Function $F$ is a homogeneous quadratic function of the velocities (which in the present instance are $\alpha$, $\beta$, $\gamma$) representing half the rate at which energy is being dissipated.

In general if $T$ be kinetic energy, $V$ potential energy, and $F$ the dissipation function, the Lagrangian equation of motion corresponding to a coordinate $\psi$ is

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right)-\frac{dT}{d\psi}+\frac{dF}{d\dot{\psi}}+\frac{dV}{d\psi}=0.$$

This equation cannot be arrived at by introducing $F$ before variation into $\delta\int(T-V)\,dt$; but if we treat the energy and dissipation functions separately and afterwards piece together the results of the variations we shall get the desired result. For, neglecting terms at the time limits,

$$\delta\int(T-V)\,dt=-\Sigma\int\left\{\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right)-\frac{dT}{d\psi}+\frac{dV}{d\psi}\right\}\delta\psi\,dt,$$

and

$$\delta\int F dt=+\Sigma\int\frac{dF}{d\dot{\psi}}\delta\psi\,dt;$$

and so the Lagrangian equation is obtained by adding the coefficient of $\delta\dot{\psi}$ under the integral in $-\delta\int F dt$ to the coefficient of $\delta\psi$ under the integral in $+\delta\int(T-V)\,dt$.

5. In the present instance the coordinates $\xi$, $\eta$, $\zeta$ are not independent, being subject to the limitation $\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0$; and so, to get the conditions of the motion, we have to add the coefficient of $\delta\alpha$ under the integral in $-\delta\int F dt$ to the coefficient of $\delta\xi$ under the integral in

$$\delta\int(T+T'-W)\,dt+\int dt\,\delta\iiint\lambda\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)dx\,dy\,dz,$$

where the notation is now that of Larmor's Report, namely $T$ and $W$ representing the kinetic or electromagnetic and the static parts respectively of the energy of the medium, and $T'$ the magneto-optic part. The introduction of $\lambda$ is the characteristic feature of the theory.

6. $F$ is given by the relation

$$2F = \iiint (Pp + Qq + Rr)\, dx\, dy\, dz,$$

and as

$$P = \frac{4\pi}{K} f \quad \text{and} \quad p = \frac{P}{\sigma} = \frac{4\pi}{\sigma K} f,$$

where $\sigma$ is specific resistance, it follows that

$$F = \frac{8\pi^2}{\sigma K^2} \iiint (f^2 + g^2 + h^2)\, dx\, dy\, dz.$$

To express this in terms of $\alpha$, $\beta$, $\gamma$, we notice that

$$4\pi \left(p + \frac{df}{dt}\right) = 4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz},$$

and

$$p = (4\pi/\sigma K) f,$$

so that

$$4\pi \left(\frac{4\pi}{\sigma K} + \frac{d}{dt}\right) f = \frac{d\gamma}{dy} - \frac{d\beta}{dz}.$$

In the case of light oscillations we assume all the variables proportional to $e^{\iota p t}$, where $\iota$ denotes $\sqrt{-1}$, and $p$ is not to be confused with the $x$ component of the conduction current; the above relation then becomes

$$f = \frac{1}{4\pi (4\pi/\sigma K + \iota p)} \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right),$$

and therefore

$$F = \frac{1}{2\sigma (4\pi/\sigma + \iota p K)^2} \iiint \left[ \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)^2 + \left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right)^2 + \left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)^2 \right] dx\, dy\, dz.$$

Varying, and integrating by parts in the usual way, we obtain

$$\delta F = \frac{1}{\sigma (4\pi/\sigma + \iota p K)^2} \iiint \left[ \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)\left(\frac{d\delta\gamma}{dy} - \frac{d\delta\beta}{dz}\right) + \dots \right] dx\, dy\, dz$$

$$= \frac{1}{\sigma (4\pi/\sigma + \iota p K)^2} \left[ \iint \left[ \left\{ n \left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) - m \left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right) \right\} \delta\alpha + \text{two similar} \right] dS \right.$$

$$\left. - \iiint \left[ \left\{ \frac{d}{dz}\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) - \frac{d}{dy}\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right) \right\} \delta\alpha + \text{two similar} \right] dx\, dy\, dz \right],$$

where $l$, $m$, $n$ are the direction cosines of the outward normal to the element $dS$ of the bounding surface, and the surface integral is taken over all bounding surfaces.

7. The value of $T$ being

$$T = \frac{\mu}{8\pi}\iiint\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right\} dx\,dy\,dz,$$

the corresponding variation is

$$\delta\int T dt = \int dt\,\frac{\mu}{4\pi}\iiint\left\{\frac{d\xi}{dt}\frac{d\delta\xi}{dt} + \frac{d\eta}{dt}\frac{d\delta\eta}{dt} + \frac{d\zeta}{dt}\frac{d\delta\zeta}{dt}\right\} dx\,dy\,dz$$

$$= -\int dt\,\frac{\mu}{4\pi}\iiint\left\{\frac{d^2\xi}{dt^2}\delta\xi + \frac{d^2\eta}{dt^2}\delta\eta + \frac{d^2\zeta}{dt^2}\delta\zeta\right\} dx\,dy\,dz,$$

terms at the time limits being neglected.

8. From the expression for $T'$ assumed in § 3 we derive

$$\delta\int T' dt = C\int dt\left[\iiint\left\{\frac{d\delta\xi}{d\theta}\frac{d}{dt}\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right) + \frac{d\delta\eta}{d\theta}\frac{d}{dt}\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right) + \frac{d\delta\zeta}{d\theta}\frac{d}{dt}\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)\right\} dx\,dy\,dz\right.$$

$$\left. - \iiint\left\{\frac{d^2\xi}{d\theta dt}\left(\frac{d\delta\zeta}{dy} - \frac{d\delta\eta}{dz}\right) + \frac{d^2\eta}{d\theta dt}\left(\frac{d\delta\xi}{dz} - \frac{d\delta\zeta}{dx}\right) + \frac{d^2\zeta}{d\theta dt}\left(\frac{d\delta\eta}{dx} - \frac{d\delta\xi}{dy}\right)\right\} dx\,dy\,dz\right]$$

$$= C\int dt\left[\iint(\alpha_0 l + \beta_0 m + \gamma_0 n)\frac{d}{dt}\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)\delta\xi\, dS + \text{two similar}\right.$$

$$+ \iint\left(m\frac{d^2\zeta}{d\theta dt} - n\frac{d^2\eta}{d\theta dt}\right)\delta\xi\, dS + \text{two similar}$$

$$\left. - 2\iiint\frac{d^2}{d\theta dt}\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)\delta\xi\, dx\,dy\,dz - \text{two similar}\right].$$

Also $\delta\int dt\iiint\lambda\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) dx\,dy\,dz$

$$= \int dt\left[\iint\lambda\,(l\delta\xi + m\delta\eta + n\delta\zeta)\, dS - \iiint\left(\frac{d\lambda}{dx}\delta\xi + \frac{d\lambda}{dy}\delta\eta + \frac{d\lambda}{dz}\delta\zeta\right) dx\,dy\,dz\right].$$

9. The static part of the energy of the medium is given by

$$W = (2\pi/K)\iiint(f^2 + g^2 + h^2)\, dx\,dy\,dz$$

$$= \frac{1}{8\pi K\,(4\pi/\sigma K + \iota p)^2}\iiint\left[\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)^2 + \left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right)^2 + \left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)^2\right] dx\,dy\,dz;$$

from which, noticing that $(\alpha, \beta, \gamma) = \iota p\,(\xi, \eta, \zeta)$, it follows that

$$\delta\int(-W)\, dt = \int dt\,\frac{p^2}{4\pi K\,(4\pi/\sigma K + \iota p)^2}\left[\iiint\left\{\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)\left(\frac{d\delta\zeta}{dy} - \frac{d\delta\eta}{dz}\right)\right.\right.$$

$$\left.\left. + \left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right)\left(\frac{d\delta\xi}{dz} - \frac{d\delta\zeta}{dx}\right) + \left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)\left(\frac{d\delta\eta}{dx} - \frac{d\delta\xi}{dy}\right)\right\} dx\,dy\,dz\right]$$

$$= \int dt\,\frac{p^2}{4\pi K\,(4\pi/\sigma K + \iota p)^2}\left[\iint\left\{n\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right) - m\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)\right\}\delta\xi\, dS + \text{two similar}\right.$$

$$\left. - \iiint\left\{\frac{d}{dz}\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right) - \frac{d}{dy}\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)\right\}\delta\xi\, dx\,dy\,dz - \text{two similar}\right].$$

10. If now we bring together the results of the last four paragraphs in the manner specified in § 5, we obtain the following expression of the conditions of the motion:—

$$
\begin{aligned}
&-\frac{\mu}{4\pi}\iiint\left(\frac{d^2\xi}{dt^2}\,\delta\xi+\frac{d^2\eta}{dt^2}\,\delta\eta+\frac{d^2\zeta}{dt^2}\,\delta\zeta\right)dx\,dy\,dz \\
&\qquad +C\left[\iint(\alpha_0 l+\beta_0 m+\gamma_0 n)\frac{d}{dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\delta\xi\,dS+\text{two similar}\right. \\
&\qquad\qquad +\iint\left(m\frac{d^2\zeta}{d\theta\,dt}-n\frac{d^2\eta}{d\theta\,dt}\right)\delta\xi\,dS+\text{two similar} \\
&\qquad\qquad \left.-2\iiint\frac{d^2}{d\theta\,dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)\delta\xi\,dx\,dy\,dz-\text{two similar}\right] \\
&\qquad +\frac{p^2}{4\pi K(4\pi/\sigma K+\iota p)^2}\left[\iint\left\{n\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-m\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}\delta\xi\,dS+\text{two similar}\right. \\
&\qquad\qquad \left.-\iiint\left\{\frac{d}{dz}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-\frac{d}{dy}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}\delta\xi\,dx\,dy\,dz-\text{two similar}\right] \\
&\qquad +\iint\lambda\,(l\,\delta\xi+m\,\delta\eta+n\,\delta\zeta)\,dS-\iiint\left(\frac{d\lambda}{dx}\,\delta\xi+\frac{d\lambda}{dy}\,\delta\eta+\frac{d\lambda}{dz}\,\delta\zeta\right)dx\,dy\,dz \\
&\qquad -\frac{1}{\sigma(4\pi/\sigma+\iota pK)^2}\left[\iint\left\{n\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)-m\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\right\}\delta\xi\,dS+\text{two similar}\right. \\
&\qquad\qquad \left.-\iiint\left\{\frac{d}{dz}\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)-\frac{d}{dy}\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\right\}\delta\xi\,dx\,dy\,dz-\text{two similar}\right] \\
&\qquad =0.
\end{aligned}
$$

Taking together the volume integrals we get the bodily equations of propagation, of the type

$$
\mu\frac{d^2\xi}{dt^2}=\frac{\iota p}{4\pi/\sigma+\iota pK}\left[\frac{d}{dz}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)-\frac{d}{dy}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right]-8\pi C\frac{d^2}{d\theta\,dt}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)-4\pi\frac{d\lambda}{dx}\quad\text{.....(i).}
$$

From the surface integrals we obtain the boundary conditions which hold good at an interface between two different media; in this case of course the integrals extend over both sides of the surface of separation. If, for the sake of simplification, we take the axis of $z$ normal to the element of surface considered, we have $l=0$, $m=0$, $n=1$. Now $\delta\xi$ and $\delta\eta$ must be continuous across the interface, and therefore so also must be the expressions which are their coefficients in the surface integral: $\delta\zeta$ is not necessarily continuous, for reasons explained by Mr. Larmor in the footnote to § 11 of his Report; he has however pointed out to me that the supposition that it is continuous is perfectly allowable and involves no inconsistency. Thus we have analytically the alternative of supposing that both $\delta\zeta$ and its coefficient in the surface integral are continuous across the interface, or on the other hand of supposing that $\delta\zeta$ is discontinuous, and that therefore its coefficient vanishes at both sides of the bounding surface. Of these the former supposition seems the more natural, but the consequences of both will be investigated.

The boundary conditions are accordingly as follows:

(I) $\xi$ and $\eta$ continuous,

(II) $\dfrac{-\iota p}{4\pi/\sigma + \iota p K}\left(\dfrac{d\xi}{dz} - \dfrac{d\zeta}{dx}\right) - 4\pi C \dfrac{d^2\eta}{d\theta\, dt} + 4\pi C\gamma_0 \dfrac{d}{dt}\left(\dfrac{d\zeta}{dy} - \dfrac{d\eta}{dz}\right)$ continuous,

(III) $\dfrac{\iota p}{4\pi/\sigma + \iota p K}\left(\dfrac{d\zeta}{dy} - \dfrac{d\eta}{dz}\right) + 4\pi C \dfrac{d^2\xi}{d\theta\, dt} + 4\pi C\gamma_0 \dfrac{d}{dt}\left(\dfrac{d\xi}{dz} - \dfrac{d\zeta}{dx}\right)$ continuous,

(IV) Either (1°) $\zeta$ continuous,

and $\lambda + C\gamma_0 \dfrac{d}{dt}\left(\dfrac{d\eta}{dx} - \dfrac{d\xi}{dy}\right)$ continuous,

or (2°) $\lambda + C\gamma_0 \dfrac{d}{dt}\left(\dfrac{d\eta}{dx} - \dfrac{d\xi}{dy}\right) = 0$ at both sides of the boundary.

## PLANE WAVES.

11. In dealing with the problem of the reflexion and refraction of plane waves at a plane surface of magnetised metal, it is convenient to take the reflecting surface as the plane $z = 0$, and the plane of incidence as the plane $y = 0$; the positive direction of the axis of $z$ is from the metal into the air.

If we suppose that the expressions representing the optical circumstances in the incident wave depend on the exponential $e^{\iota(lx+mz+pt)}$, then those which represent the circumstances in a corresponding reflected or refracted wave must, in so far as they involve $x$ and $t$, depend on the exponential $e^{\iota(lx+pt)}$. Hence the most general assumption that we can make about a reflected or refracted wave is that its rotational part depends on one or more exponentials of the type $e^{\iota(lx+m'z+pt)}$, and its condensational part on others of similar form, say $e^{\iota(lx+m''z+pt)}$. In fact, for such a wave

$$\left.\begin{aligned}
\xi &= \Sigma A e^{\iota(lx+m'z+pt)} + \Sigma d\phi/dx \\
\eta &= \Sigma B e^{\iota(lx+m'z+pt)} + \Sigma d\phi/dy \\
\zeta &= \Sigma\,(-l/m')\,A e^{\iota(lx+m'z+pt)} + \Sigma d\phi/dz \\
\phi &= \mathfrak{A} e^{\iota(lx+m''z+pt)} \\
\lambda &= \Sigma L e^{\iota(lx+m''z+pt)}
\end{aligned}\right\} \quad \text{(v)},$$

where $A$, $B$, $\mathfrak{A}$, and $L$ are constants, real or complex.

The form of $\zeta$ has been so chosen that the rotational parts of $\xi$, $\eta$, $\zeta$ satisfy the condition

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0 \quad \text{(vi)};$$

the substitution of their irrotational parts in this equation leads at once to the condition $l^2 + m''^2 = 0$, which shews that there are only two possible values of $m''$, namely $m'' = +\iota l$ and $m'' = -\iota l$. The corresponding exponentials are $e^{-lz}e^{\iota(lx+pt)}$ and $e^{+lz}e^{\iota(lx+pt)}$, of which the former can occur only in a reflected, the latter only in a refracted wave, an amplitude which increases without limit in the direction of propagation being impossible.

The parts of $(\xi, \eta, \zeta)$ corresponding to different exponentials are really different waves, travelling each with its own velocity; the above condition shews that the velocity of propagation of either of the irrotational waves is infinitely great, since its square is equal to $p^2/(l^2+m''^2)$.

12. In order to obtain further information about the various constants which occur in the assumed expressions for $\xi$, $\eta$, $\zeta$, we substitute these values in the equations of propagation. In the case of the irrotational terms this leads to the relation

$$p^2\mu\mathfrak{A} = 4\pi L \quad\text{.....(vii).}$$

From the rotational terms we get, for each value of $m'$, two relations which readily reduce to the form

$$\left.\begin{aligned} A\left\{\mu\iota p + \frac{l^2+m'^2}{4\pi/\sigma + \iota pK}\right\} &= -8\pi C(\alpha_0 l + \gamma_0 m')\, m'B, \\ B\left\{\mu\iota p + \frac{l^2+m'^2}{4\pi/\sigma + \iota pK}\right\} &= +8\pi C(\alpha_0 l + \gamma_0 m')\,\frac{l^2+m'^2}{m'}A \end{aligned}\right\} \quad\text{.....(viii).}$$

Eliminating from these the ratio $B/A$, we get

$$\mu\iota p + \frac{l^2+m'^2}{4\pi/\sigma + \iota pK} = \pm\,\iota\,.\,8\pi C(\alpha_0 l + \gamma_0 m')\,(l^2+m'^2)^{\frac{1}{2}} \quad\text{.....(ix),}$$

an equation which determines $m'$. In this, to avoid ambiguity, we shall make the convention that $(l^2+m'^2)^{\frac{1}{2}}$, or $\omega'$, as it may for brevity be called, is a complex whose imaginary part is negative. As the equation is a quartic, there are four possible values of $m'$; and if we neglect the second and higher powers of $C$, which for all media is an exceedingly small quantity, these four values are found to be of the form $+m_1$, $-m_1$, $+m_2$, $-m_2$, of which the two former correspond to the positive, the two latter to the negative sign on the right-hand side of the equation in $m'$. We complete the definition by the supposition that of the complexes $+m_1$ and $-m_1$ the former is that which has its imaginary part negative, and $+m_2$ is chosen in the same way; as a matter of fact it is found that $m_1$ and $m_2$ so defined have their real parts positive.

If $(A_1, B_1)$, $(A_1', B_1')$, $(A_2, B_2)$, $(A_2', B_2')$ be the pairs of values of $(A, B)$ corresponding to the roots $+m_1$, $-m_1$, $+m_2$, $-m_2$ respectively, and if we make the abbreviations

$$l^2 + m_1^2 \equiv \omega_1^2, \quad l^2 + m_2^2 \equiv \omega_2^2 \quad\text{.....(x),}$$

either of the above relations between $A$ and $B$ readily yields the following

$$\left.\begin{aligned} A_1\iota\omega_1 &= -B_1 m_1, \quad A_1'\iota\omega_1 = +B_1' m_1 \\ A_2\iota\omega_2 &= +B_2 m_2, \quad A_2'\iota\omega_2 = -B_2' m_2 \end{aligned}\right\} \quad\text{.....(xi).}$$

The consideration that a wave whose amplitude increases continually in the direction of propagation cannot occur, indicates that in the problem of reflection the reflected waves involve only those exponentials corresponding to $m' = -m_1$ and $m' = -m_2$, while the refracted waves involve only those corresponding to $m' = +m_1$ and $m' = +m_2$.

13. In the particular case when $\alpha_0$ and $\gamma_0$ are zero, $m_1$ and $m_2$ are equal and their common value may be denoted by $M$; the corresponding common value of $\omega_1$ and $\omega_2$ may be called $\Omega$. To evaluate these quantities we have only to change the right-hand side of equation (ix) to zero, when we find

$$\left.\begin{aligned} \Omega^2 &= -\mu\iota p\,(4\pi/\sigma + \iota pK), \\ \text{whence of course} \quad M^2 &= -l^2 - \mu\iota p\,(4\pi/\sigma + \iota pK) \end{aligned}\right\} \quad \text{(xii)}.$$

Returning to the general case, we see that $m_1$ and $m_2$ differ from $M$, and $\omega_1$ and $\omega_2$ differ from $\Omega$, by quantities which have $C$ for a factor and which are therefore small of the first order compared with $M$ and $\Omega$. If we neglect small quantities of the second order we may substitute $\Omega$ for $\omega'$ and $M$ for $m'$ on the right-hand side of equation (ix), which may therefore be written in the form

$$\frac{\omega'^2 - \Omega^2}{4\pi/\sigma + \iota pK} = \pm\,\iota\,.\,8\pi C\,(\alpha_0 l + \gamma_0 M)\,\Omega.$$

On introducing the abbreviation

$$\begin{aligned} \varpi &= -\iota\,.\,8\pi C\,(\alpha_0 l + \gamma_0 M)\,(4\pi/\sigma + \iota pK)/\Omega, \\ &= (8\pi\Omega/\mu p)\,C\,(\alpha_0 l + \gamma_0 M) \quad \text{(xiii)}, \end{aligned}$$

equation (ix) further reduces to

$$\omega'^2 - \Omega^2 = \mp\,\varpi\Omega^2 \quad \text{(xiv)}.$$

Hence we have $\quad \omega_1^2 = (1-\varpi)\,\Omega^2, \quad \omega_2^2 = (1+\varpi)\,\Omega^2 \quad$ (xv),

and therefore also $\quad m_1^2 = M^2 - \varpi\Omega^2, \quad m_2^2 = M^2 + \varpi\Omega^2 \quad$ (xvi);

and as $\varpi$ is small, these lead to

$$\left.\begin{aligned} \omega_1 &= (1 - \tfrac{1}{2}\varpi)\,\Omega, & \omega_2 &= (1 + \tfrac{1}{2}\varpi)\,\Omega, \\ m_1 &= \left(1 - \frac{1}{2}\frac{\Omega^2}{M^2}\varpi\right)M, & m_2 &= \left(1 + \frac{1}{2}\frac{\Omega^2}{M^2}\varpi\right)M \end{aligned}\right\} \quad \text{(xvii)},$$

expressions which will prove exceedingly useful in the subsequent analysis.

14. For a medium in which there is no magneto-optic effect the relations (xi) do not hold good. In fact, as for such a medium $C$ is zero, the equations (viii) and (ix) all reduce to the same form, namely

$$\mu\iota p + \frac{l^2 + m'^2}{4\pi/\sigma + \iota pK} = 0,$$

and the ratio $A/B$ is left quite arbitrary. If, as in the case of air, the medium is also a non-conductor, $\sigma$ is infinitely great, and equation (ix) assumes the form

$$l^2 + m'^2 - p^2\mu K = 0,$$

which expresses the fact that, if $V$ be the velocity of propagation of light, $V^2 = (\mu K)^{-1}$. There are two values of $m'$ of the forms $+m$ and $-m$.

## THE REFLEXION PROBLEM.

15. On passing now to a more detailed investigation of the problem of magnetic reflexion, the preceding paragraphs justify the representation of the optical circumstances *in the air* by the following expressions:—

$$\xi = A_0 e^{\iota(lx+mz+pt)} + A e^{\iota(lx-mz+pt)} + d\phi/dx,$$

$$\eta = B_0 e^{\iota(lx+mz+pt)} + B e^{\iota(lx-mz+pt)} + d\phi/dy,$$

$$\zeta = -(l/m) A_0 e^{\iota(lx+mz+pt)} + (l/m) A e^{\iota(lx-mz+pt)} + d\phi/dz,$$

$$\phi = \mathfrak{A} e^{-lz} e^{\iota(lx+pt)}, \qquad \lambda = (p^2/4\pi) \mathfrak{A} e^{-lz} e^{\iota(lx+pt)},$$

wherein $A_0$, $B_0$ represent the incident wave and $A$, $B$ the reflected wave; the system of units being the electromagnetic, $\mu$ for air is equal to unity.

*In the metal* the refracted light may, in accordance with § 12, be represented by

$$\xi = A_1 e^{\iota(lx+m_1z+pt)} + A_2 e^{\iota(lx+m_2z+pt)} + d\phi/dx,$$

$$\eta = -\iota(\omega_1/m_1) A_1 e^{\iota(lx+m_1z+pt)} + \iota(\omega_2/m_2) A_2 e^{\iota(lx+m_2z+pt)} + d\phi/dy,$$

$$\zeta = -(l/m_1) A_1 e^{\iota(lx+m_1z+pt)} - (l/m_2) A_2 e^{\iota(lx+m_2z+pt)} + d\phi/dz,$$

$$\phi = \mathfrak{A}' e^{+lz} e^{\iota(lx+pt)}, \qquad \lambda = (p^2\mu/4\pi) \mathfrak{A}' e^{+lz} e^{\iota(lx+pt)}.$$

Getting rid of the $\phi$'s, the values of $\xi$, $\eta$, $\zeta$, and $\lambda$ in the two media may with advantage be rewritten as follows:—

In the air

$$\left.\begin{aligned}
\xi &= A_0 e^{\iota(lx+mz+pt)} + A e^{\iota(lx-mz+pt)} + \iota l \mathfrak{A} e^{-lz} e^{\iota(lx+pt)},\\
\eta &= B_0 e^{\iota(lx+mz+pt)} + B e^{\iota(lx-mz+pt)},\\
\zeta &= -(l/m) A_0 e^{\iota(lx+mz+pt)} + (l/m) A e^{\iota(lx-mz+pt)} - l \mathfrak{A} e^{-lz} e^{\iota(lx+pt)},\\
\lambda &= (p^2/4\pi) \mathfrak{A} e^{-lz} e^{\iota(lx+pt)}
\end{aligned}\right\} \text{.....(xviii).}$$

In the metal

$$\left.\begin{aligned}
\xi &= A_1 e^{\iota(lx+m_1z+pt)} + A_2 e^{\iota(lx+m_2z+pt)} + \iota l \mathfrak{A}' e^{lz} e^{\iota(lx+pt)},\\
\eta &= -\iota(\omega_1/m_1) A_1 e^{\iota(lx+m_1z+pt)} + \iota(\omega_2/m_2) A_2 e^{\iota(lx+m_2z+pt)},\\
\zeta &= -(l/m_1) A_1 e^{\iota(lx+m_1z+pt)} - (l/m_2) A_2 e^{\iota(lx+m_2z+pt)} + l \mathfrak{A}' e^{lz} e^{\iota(lx+pt)},\\
\lambda &= (p^2\mu/4\pi) \mathfrak{A}' e^{lz} e^{\iota(lx+pt)}
\end{aligned}\right\} \text{.....(xix).}$$

15 *a*. Let us first consider the second of the alternative hypotheses referred to at the end of § 10, namely that which supposes $\zeta$ discontinuous.

The boundary condition (IV, 2°) must now be used, and from it it appears that, since in air $C$ is zero, $\mathfrak{A}$ must also be zero. A slight simplification thus takes place in the expressions representing the optical circumstances in the air. Substituting these expressions in the boundary conditions we obtain:—

(I) From the continuity of $\xi$ and $\eta$

$$A_0 + A = A_1 + A_2 + \iota l \mathfrak{A}' \quad \text{.....(xx)},$$

$$B_0 + B = -\iota(\omega_1/m_1) A_1 + \iota(\omega_2/m_2) A_2 \quad \text{.....(xxi)}.$$

(II) From condition (II)

$$-\frac{1}{K}\iota\frac{l^2+m^2}{m}(A_0 - A) = \frac{-\iota p}{4\pi/\sigma + \iota p K'}\iota\left\{\frac{l^2+m_1^2}{m_1}A_1 + \frac{l^2+m_2^2}{m_2}A_2\right\}$$
$$+ 4\pi C p\{(\alpha_0 l + \gamma_0 m_1)(-\iota\omega_1/m_1)A_1 + (\alpha_0 l + \gamma_0 m_2)(+\iota\omega_2/m_2)A_2\}$$
$$+ 4\pi C\gamma_0 \iota p(-\omega_1 A_1 + \omega_2 A_2) \quad \text{.....(xxii)}.$$

(III) From condition (III)

$$\frac{1}{K}(-\iota m)(B_0 - B) = \frac{\iota p}{4\pi/\sigma + \iota p K'}(-\omega_1 A_1 + \omega_2 A_2)$$
$$- 4\pi C p\{(\alpha_0 l + \gamma_0 m_1)A_1 + (\alpha_0 l + \gamma_0 m_2)A_2 + (\iota\alpha_0 l + \gamma_0 l) l\mathfrak{A}'\}$$
$$+ 4\pi C\gamma_0 \iota p\left\{\iota\frac{l^2+m_1^2}{m_1}A_1 + \iota\frac{l^2+m_2^2}{m_2}A_2\right\} \quad \text{.....(xxiii)}.$$

(IV) From condition (IV, 2°)

$$(p^2\mu/4\pi)\mathfrak{A}' - C\gamma_0 p l\{-\iota(\omega_1/m_1)A_1 + \iota(\omega_2/m_2)A_2\} = 0 \quad \text{.....(xxiv)}.$$

In these results the specific inductive capacity of the metal is denoted by $K'$ to distinguish it from that of air. Equation (xxiv) shows that when $\gamma_0$ is zero so also is $\mathfrak{A}'$, so that when the reflexion is equatorial there is no condensational wave; it also shews that $\mathfrak{A}'$ is small of the first order compared with $A_1$ or $A_2$, and may therefore be omitted from equation (xxiii).

If we eliminate $\mathfrak{A}'$ from these five equations, and, neglecting small quantities of the second order, substitute in terms containing the factor $C$ the first approximations $\Omega$ for $\omega_1$ or $\omega_2$ and $M$ for $m_1$ or $m_2$, we obtain

$$A_0 + A = A_1 + A_2 + \frac{4\pi C\gamma_0}{p\mu} l^2 \frac{\Omega}{M}(A_1 - A_2) \quad \text{.....(xxv)},$$

$$\iota(B_0 + B) = (\omega_1/m_1)A_1 - (\omega_2/m_2)A_2 \quad \text{.....(xxvi)},$$

$$\frac{1}{K}\frac{\omega^2}{m}(A_0 - A) = \frac{\iota p}{4\pi/\sigma + \iota p K'}\left\{\frac{\omega_1^2}{m_1}A_1 + \frac{\omega_2^2}{m_2}A_2\right\}$$
$$+ 4\pi C p\frac{\Omega}{M}(\alpha_0 l + 2\gamma_0 M)(A_1 - A_2) \quad \text{.....(xxvii)},$$

$$\frac{1}{K}m\iota(B_0 - B) = \frac{\iota p}{4\pi/\sigma + \iota p K'}\{\omega_1 A_1 - \omega_2 A_2\}$$
$$+ 4\pi C p(\alpha_0 l + \gamma_0 M)(A_1 + A_2) + 4\pi C p\gamma_0\frac{\Omega^2}{M}(A_1 + A_2) \quad \text{.....(xxviii)}.$$

On substituting the values of $m_1$, $m_2$, $\omega_1$, $\omega_2$ from § 13 and remembering result (xii), these become

$$\left.\begin{aligned}
A_0+A &= A_1+A_2+\frac{4\pi C\gamma_0}{p\mu}\,l^2\frac{\Omega}{M}(A_1-A_2),\\
\iota(B_0+B) &= \frac{\Omega}{M}(A_1-A_2)+\frac{4\pi C}{p\mu}\cdot\frac{l^2\Omega^2}{M^3}(\alpha_0 l+\gamma_0 M)(A_1+A_2),\\
\frac{1}{K}\frac{\omega^2}{m}(A_0-A) &= \frac{p^2\mu}{M}(A_1+A_2)+4\pi Cp\frac{\Omega}{M^3}(\alpha_0 l^3+\gamma_0\Omega^2 M)(A_1-A_2),\\
\frac{1}{K}m\iota(B_0-B) &= \frac{p^2\mu}{\Omega}(A_1-A_2)+4\pi Cp\gamma_0\frac{\Omega^2}{M}(A_1+A_2)
\end{aligned}\right\}\text{.....(xxix).}$$

From the first and second of these equations we obtain, to first order of small quantities,

$$A_1+A_2 = A_0+A-\frac{4\pi C\gamma_0}{p\mu}\,l^2\iota(B_0+B),$$

$$\frac{\Omega}{M}(A_1-A_2) = \iota(B_0+B)-\frac{4\pi C}{p\mu}\cdot\frac{l^2\Omega^2}{M^3}(\alpha_0 l+\gamma_0 M)(A_0+A);$$

and if we substitute these values in the second and third equations, and remember that $1/K = V^2 = p^2/\omega^2$, we get relations which readily reduce to

$$\left.\begin{aligned}
\frac{1}{m}(A_0-A) &= \frac{\mu}{M}(A_0+A)+\frac{4\pi C}{pM^2}(\alpha_0 l^3+\gamma_0 M^3)\,\iota(B_0+B),\\
\frac{m}{\omega^2}\iota(B_0-B) &= \mu\frac{M}{\Omega^2}\iota(B_0+B)-\frac{4\pi C}{pM^2}(\alpha_0 l^3-\gamma_0 M^3)(A_0+A)
\end{aligned}\right\}\text{.....(xxx).}$$

Solving these for $A$ and $B$ we have

$$\left.\begin{aligned}
A &= \left[\left(\frac{1}{m}-\frac{\mu}{M}\right)\left(\frac{m}{\omega^2}+\mu\frac{M}{\Omega^2}\right)A_0-\frac{8\pi Cm}{pM^2\omega^2}(\alpha_0 l^3+\gamma_0 M^3)\,\iota B_0\right]\Big/\left(\frac{1}{m}+\frac{\mu}{M}\right)\left(\frac{m}{\omega^2}+\mu\frac{M}{\Omega^2}\right)\\
\iota B &= \left[\frac{8\pi C}{pM^2 m}(\alpha_0 l^3-\gamma_0 M^3)A_0+\left(\frac{1}{m}+\frac{\mu}{M}\right)\left(\frac{m}{\omega^2}-\mu\frac{M}{\Omega^2}\right)\iota B_0\right]\Big/\left(\frac{1}{m}+\frac{\mu}{M}\right)\left(\frac{m}{\omega^2}+\mu\frac{M}{\Omega^2}\right)
\end{aligned}\right\}\text{...(xxxi),}$$

which, since $A$ and $B$ specify the reflected light, constitute the complete formal solution of the problem of metallic reflexion.

15 *b*. Turning now to the consideration of the first of the alternative hypotheses of § 10, namely that which supposes $\zeta$ continuous, we proceed exactly as before, save only that we use the boundary conditions (IV, 1°) instead of (IV, 2°). In this case, of course, $\mathfrak{A}$ is not zero.

Substituting the full expressions (xviii) and (xix) in the boundary conditions we obtain:—

(I) From the continuity of $\xi$, $\eta$, and $\zeta$

$$A_0+A+\iota l\mathfrak{A} = A_1+A_2+\iota l\mathfrak{A}',$$

$$B_0+B = -\iota(\omega_1/m_1)A_1+\iota(\omega_2/m_2)A_2,$$

$$-\frac{1}{m}(A_0-A)-\mathfrak{A} = -\frac{1}{m_1}A_1-\frac{1}{m_2}A_2+\mathfrak{A}'.$$

(II) From condition (II)

$$-\frac{1}{K}\iota\frac{l^2+m^2}{m}(A_0-A)=\frac{-\iota p}{4\pi/\sigma+\iota pK'}\iota\left\{\frac{l^2+m_1^2}{m_1}A_1+\frac{l^2+m_2^2}{m_2}A_2\right\}$$
$$+4\pi Cp\{(\alpha_0 l+\gamma_0 m_1)(-\iota\omega_1/m_1)A_1+(\alpha_0 l+\gamma_0 m_2)(\iota\omega_2/m_2)A_2\}$$
$$+4\pi C\gamma_0\iota p(-\omega_1 A_1+\omega_2 A_2).$$

(III) From condition (III)

$$\frac{1}{K}(-\iota m)(B_0-B)=\frac{\iota p}{4\pi/\sigma+\iota pK'}(-\omega_1 A_1+\omega_2 A_2)$$
$$-4\pi Cp\{(\alpha_0 l+\gamma_0 m_1)A_1+(\alpha_0 l+\gamma_0 m_2)A_2+(\iota\alpha_0 l+\gamma_0 l)\,l\mathfrak{A}'\}$$
$$+4\pi C\gamma_0\iota p\left\{\iota\frac{l^2+m_1^2}{m_1}A_1+\iota\frac{l^2+m_2^2}{m_2}A_2\right\}.$$

(IV) From condition (IV, 1°)

$$(p^2/4\pi)\,\mathfrak{A}=(p^2\mu/4\pi)\,\mathfrak{A}'-C\gamma_0 pl\{-\iota(\omega_1/m_1)A_1+\iota(\omega_2/m_2)A_2\}.$$

If we substitute in these equations the values of $m_1$, $m_2$, $\omega_1$, $\omega_2$ from § 13, replace $(4\pi/\sigma+\iota pK')$ by $\iota\Omega^2/\mu p$ in virtue of (xii), and omit small quantities of the second and higher orders, we obtain :—

$$A_0+A+\iota l\mathfrak{A}=A_1+A_2+\iota l\mathfrak{A}',$$
$$\iota(B_0+B)=\frac{\Omega}{M}(A_1-A_2)+\frac{4\pi C}{p\mu}\frac{l^2\Omega^2}{M^3}(\alpha_0 l+\gamma_0 M)(A_1+A_2),$$
$$\frac{1}{m}(A_0-A)+\mathfrak{A}=\frac{1}{M}(A_1+A_2)+\frac{4\pi C}{p\mu}\frac{\Omega^2}{M^3}(\alpha_0 l+\gamma_0 M)(A_1-A_2),$$
$$\frac{1}{K}\frac{\omega^2}{m}(A_0-A)=\frac{p^2\mu}{M}(A_1+A_2)+4\pi Cp\frac{\Omega}{M^3}(\alpha_0 l^2+\gamma_0\Omega^2 M)(A_1-A_2),$$
$$\frac{1}{K}m\iota(B_0-B)=\frac{p^2\mu}{\Omega}(A_1-A_2)+4\pi Cp\gamma_0\frac{\Omega^2}{M}(A_1+A_2)+4\pi Cpl^2(\iota\alpha_0+\gamma_0)\,\mathfrak{A}',$$
$$\iota p^2\mathfrak{A}=\iota p^2\mu\mathfrak{A}'-4\pi Cp\gamma_0\frac{l\Omega}{M}(A_1-A_2).$$

Solving the second and third of these for $(A_1+A_2)$ and $(A_1-A_2)$, substituting the values so obtained in the others, and remembering that $1/K=p^2/\omega^2$, we get :—

$$(M-\iota l)\,\mathfrak{A}+\iota l\mathfrak{A}'=A_0+A-\frac{M}{m}(A_0-A)+\frac{4\pi C}{p\mu}\frac{\Omega^2}{M}(\alpha_0 l+\gamma_0 M)\,\iota(B_0+B),$$
$$p^2\mu\mathfrak{A}=-p^2\frac{\mu-1}{m}(A_0-A)+4\pi Cp\alpha_0 l\iota(B_0+B),$$
$$-4\pi Cp\frac{1}{M}(\alpha_0 l^2-\gamma_0 M^2)\,\mathfrak{A}+4\pi Cpl^2(\iota\alpha_0+\gamma_0)\,\mathfrak{A}'$$
$$=\frac{p^2 m}{\omega^2}\iota(B_0-B)-p^2\mu\frac{M}{\Omega^2}\iota(B_0+B)+4\pi Cp\frac{1}{M}(\alpha_0 l^2-\gamma_0 M^2)\frac{1}{m}(A_0-A),$$
$$\iota p^2\mathfrak{A}-\iota p^2\mu\mathfrak{A}'=-4\pi Cp\gamma_0 l\iota(B_0+B).$$

When we assume, as it is usual to do, that for magnetic forces alternating as rapidly as those in light waves, the magnetic permeability is unity, the form of these equations becomes simpler; and the elimination from them of $\mathfrak{A}$ and $\mathfrak{A}'$ which are now seen to be small of the first order, leads to:—

$$A_0 + A - \frac{M}{m}(A_0 - A) + \frac{4\pi C}{p}\frac{1}{M}(\alpha_0 l^2 + \gamma_0 M^2)\,\iota\,(B_0 + B) = 0,$$

$$\frac{m}{\omega^2}\iota\,(B_0 - B) - \frac{M}{\Omega^2}\iota\,(B_0 + B) + \frac{4\pi C}{p}\frac{1}{M}(\alpha_0 l^2 - \gamma_0 M^2)\frac{1}{m}(A_0 - A) = 0.$$

Solving for $A$ and $B$, we get

$$A = \left[\left(\frac{1}{m} - \frac{1}{M}\right)\left(\frac{m}{\omega^2} + \frac{M}{\Omega^2}\right)A_0 - \frac{8\pi C m}{pM^2\omega^2}(\alpha_0 l^2 + \gamma_0 M^2)\,\iota B_0\right] \Big/ \left(\frac{1}{m} + \frac{1}{M}\right)\left(\frac{m}{\omega^2} + \frac{M}{\Omega^2}\right),$$

$$\iota B = \left[\frac{8\pi C}{pM^2 m}(\alpha_0 l^2 - \gamma_0 M^2)\,A_0 + \left(\frac{1}{m} + \frac{1}{M}\right)\left(\frac{m}{\omega^2} - \frac{M}{\Omega^2}\right)\iota B_0\right] \Big/ \left(\frac{1}{m} + \frac{1}{M}\right)\left(\frac{m}{\omega^2} + \frac{M}{\Omega^2}\right).$$

Now the expressions here obtained for $A$ and $B$ are identical with those of equations (xxxi) when in the latter $\mu$ is, as usual, put equal to unity. Thus it appears that the alternative hypotheses as to boundary conditions discussed above lead to precisely the same results, and it is a matter of indifference which we adopt. The subsequent calculations apply equally well to the two views.

16. The value of $\Omega$ is determined by the consideration that $\Omega^2/\omega^2 = R^2 e^{2\iota\alpha}$, where $Re^{\iota\alpha}$ is the quasi refractive index of the metal*. The quantities $R$ and $\alpha$ are connected by the relations

$$R^2 \cos 2\alpha = n^2(1 - k^2), \qquad R^2 \sin 2\alpha = -2n^2 k,$$

with Drude's optic constants, whose values for different metals are quoted in Thomson's *Recent Researches*†. The value of $M$ is obtained from that of $\Omega$ by the relation $M^2 = \Omega^2 - l^2$, and it will be convenient to denote $M/\omega$ by the symbol $\mathfrak{M}$, so that

$$\mathfrak{M}^2 = R^2 e^{2\iota\alpha} - l^2/\omega^2 \qquad \text{(xxxii)}.$$

If $i$ be the angle of incidence, and if we suppose the direction of the incident light to lie in the quadrant between the positive direction of the axis of $x$ and the negative direction of the axis of $z$, then, $\omega$ and $p$ being assumed essentially positive, we have

$$l = -\omega \sin i, \qquad m = +\omega \cos i \qquad \text{(xxxiii)};$$

of course $p = V\omega$, and $\omega = 2\pi/\lambda$ where $\lambda$ is the wave length of the light in air.

We shall also put $\mu = 1$, as it is usually taken for granted that for magnetic forces alternating as rapidly as those in light waves the magnetic permeability is unity.

* J. J. Thomson, *Recent Researches*, p. 419.

† Drude, *Wied. Ann.* XXXIX. p. 481. For the constants of Cobalt see Drude, *Wied. Ann.* XLVI. p. 407.

When these substitutions are made, equations (xxxi) assume the following form:—

$$\left.\begin{aligned} A &= \frac{(\mathfrak{M}-\cos i)(\mathfrak{M}R^{-2}e^{-2\iota a}+\cos i)A_0+16\pi^2CV^{-1}\lambda^{-1}(\cos^2 i/\mathfrak{M})(\alpha_0\sin^2 i-\gamma_0\mathfrak{M}^2)\iota B_0}{(\mathfrak{M}+\cos i)(\mathfrak{M}R^{-2}e^{-2\iota a}+\cos i)} \\ \iota B &= \frac{-16\pi^2CV^{-1}\lambda^{-1}\mathfrak{M}^{-1}(\alpha_0\sin^2 i+\gamma_0\mathfrak{M}^2)A_0-(\mathfrak{M}+\cos i)(\mathfrak{M}R^{-2}e^{-2\iota a}-\cos i)\iota B_0}{(\mathfrak{M}+\cos i)(\mathfrak{M}R^{-2}e^{-2\iota a}+\cos i)} \end{aligned}\right\} \quad \text{.....(xxxv).}$$

It is to be noticed that these expressions contain only one undetermined constant, namely $C$; this we may assume complex, of the form $C_0e^{\iota x}$, where $x$ is defined as lying between $-90°$ and $+90°$, and $C_0$ may be positive or negative. The test of the theory consists in ascertaining whether it is possible, by attributing, for each metal, suitable values to $C_0$ and $x$, to derive from these formulae results in numerical agreement with those arrived at experimentally. Should the values of these constants indicated by the different experimental observations or series of observations prove to be the same, the theory may be regarded as offering a satisfactory account of the phenomena; but if the different series of observations point to considerably different values of the constants, it must be concluded that the theory is at fault.

## THE KERR EXPERIMENTS.

17. In the original experiments on magnetic reflexion, namely those of Dr Kerr, the incident light was plane polarised; and observations were made, for various incidences, of the angle between the direction of the major axis of the ellipse of polarisation of the reflected light and that direction which it would have had if there had been no magnetisation. We may denote this angle by $\theta$; it is the rotation required to bring the analyser from the position of extinction or greatest darkness before magnetisation of the mirror into the corresponding position after the magnetising current has been made, and it is to be reckoned positive when, as seen by the observer, it takes place in the direction contrary to that of the hands of a watch.

When the incident light is polarised either in or perpendicularly to the plane of incidence the theoretical value of $\theta$, which in these particular cases we shall denote by $\theta_i$ and $\theta_p$ respectively, is very simply obtained from the formulae (xxxiv). For, as in the former case $B_0=0$, and in the latter $A_0=0$, we have

$$\begin{aligned} \theta_i &= \text{real part of } \{B\cos i/A\}_{B_0=0} \\ &= \text{real part of } \frac{16\pi^2\iota C_0e^{\iota x}V^{-1}\lambda^{-1}\mathfrak{M}^{-1}(\alpha_0\sin^2 i+\gamma_0\mathfrak{M}^2)\cos i}{(\mathfrak{M}-\cos i)(\mathfrak{M}R^{-2}e^{-2\iota a}+\cos i)} \quad \text{.....(xxxv),} \\ \theta_p &= -\text{real part of } \{A/B\cos i\}_{A_0=0} \\ &= \text{real part of } \frac{16\pi^2\iota C_0e^{\iota x}V^{-1}\lambda^{-1}(\cos i/\mathfrak{M})(\alpha_0\sin^2 i-\gamma_0\mathfrak{M}^2)}{(\mathfrak{M}+\cos i)(\mathfrak{M}R^{-2}e^{-2\iota a}-\cos i)} \quad \text{.....(xxxvi).} \end{aligned}$$

When the reflexion is equatorial, so that $\gamma_0=0$, it was observed by Kerr that $\theta_p$ vanishes when the angle of incidence is about 75°, the mirror being of iron. Later

experiments point to a rather greater value of the angle of incidence for which $\theta_p$ is zero, the results obtained by different observers being as follows:

| | | | |
|---|---|---|---|
| Kerr | 75° | Sissingh | 80° |
| Righi | 78° 54′ | Kundt | 80° to 82° |
| Drude | 79° | | |

Now when $\theta_p$ is zero, the complex of which it is the real part must have its vector angle equal to an odd number of right angles; and from (xxxvi) we see that this vector angle is $\phi$, where

$\phi \equiv x + 90° -$ the sum of the vector angles of $\mathfrak{M}$, $(\mathfrak{M} + \cos i)$, and $(\mathfrak{M}R^{-2}e^{-2ia} - \cos i)$.

For any assigned value of $i$ the values of these three vector angles may be calculated from relation (xxxii), utilising the tabulated values of the optical constants for iron; the calculations, though tedious, are quite straightforward. The following table shews values of $\phi$ obtained in this way:

| Angle of Incidence | 75° | 78° 54′ | 80° |
|---|---|---|---|
| $\phi$ | $x + 80° 18'$ | $x + 98° 1'$ | $x + 103° 36'$ |

Hence the theory agrees with Kerr's observation provided $x + 80° 18' = 90°$, or $x = +9° 42'$; but according to Righi's result $x = -8° 1'$, and according to Sissingh's $x = -13° 36'$. Thus the uncertainty of the observations renders it impossible to draw from them any definite conclusion as to what value ought to be attributed to the constant $x$ in the theory. We can, however, determine the sign of $C_0$; for all the observers agree that, for angles of incidence less than 75°, $\theta_p$ has the same sign as $\alpha_0$*; but for such incidences $\cos\phi$ is positive, and accordingly $C_0$ must also be positive.

When the reflexion is polar, so that $\alpha_0 = 0$, the mirror still being of iron, it was observed by Kerr that $\theta_p$ has the sign opposite to that of $\gamma_0$ for all angles of incidence. In order that this should be in accordance with the theory it is necessary that, if

$\phi' \equiv x + 90° +$ vector angle of $\mathfrak{M}^2 -$ the sum of the vector angles of

$(\mathfrak{M} + \cos i)$ and $(\mathfrak{M}R^{-2}e^{-2ia} - \cos i)$,

$C_0 \cos\phi'$ should be positive for all angles of incidence. Now the values of $\phi'$ lie between $x + 219° 43'$ corresponding to $i = 0$, and $x + 342° 40'$ corresponding to $i = 90°$; so that either $C_0$ is positive and $x$ between 50° 17′ and 107° 20′, or $C_0$ is negative and $x$ between $-72° 40'$ and $-129° 43'$. This experiment was repeated by Kundt, who found that $\theta_p$ vanishes, changing sign, when $i$ is about 82°; the corresponding value of $\phi'$ is $x + 312° 37'$, and as the cosine of this is to vanish, either $x = +137° 23'$ and $C_0$ is positive, or $x = -42° 37'$ and $C_0$ is negative: as by definition $x$ is numerically less than 90°, only the latter of these values is admissible.

* Kerr, *Phil. Mag.*, March 1878, p. 166.

With regard to $\theta_i$, the experiments shew that in the case of polar reflexion it has always the sign opposite to that of $\gamma_0$, and that in the case of equatorial reflexion it has always the sign opposite to that of $\alpha_0$. Hence if we define $\phi''$ and $\phi'''$ by the relations

$$\phi'' \equiv x + 90^\circ + \text{vector angle of } \mathfrak{M}^2 - \text{the sum of the vector angles of } (\mathfrak{M} - \cos i) \text{ and } (\mathfrak{M}R^{-2}e^{-2\mu a} + \cos i),$$

$$\phi''' \equiv x + 90^\circ - \text{the sum of the vector angles of } \mathfrak{M}, (\mathfrak{M} - \cos i), \text{ and } (\mathfrak{M}R^{-2}e^{-2\mu a} + \cos i),$$

the theory requires that $C_0 \cos\phi''$ and $C_0 \cos\phi'''$ shall be negative for all angles of incidence. Now the values of $\phi''$ lie between $x + 39^\circ\, 43'$ corresponding to $i = 0^\circ$, and $x - 17^\circ\, 20'$ corresponding to $i = 90^\circ$; and the values of $\phi'''$ lie between $x + 200^\circ\, 43'$ corresponding to $i = 0^\circ$, and $x + 148^\circ\, 44'$ corresponding to $i = 90^\circ$. And therefore, when we remember that $x$ has by definition been restricted to be numerically less than $90^\circ$, it appears from the first condition that $C_0$ must be negative and $x$ somewhere between $-72^\circ\, 40'$ and $+50^\circ\, 17'$; while the second condition indicates that $C_0$ must be positive, and $x$ somewhere between $-58^\circ\, 44'$ and $+69^\circ\, 17'$.

Thus it appears that there are very serious discrepancies in the values of $C_0$ and $x$ indicated by the four original Kerr experiments for iron.

## THE EXPERIMENTS OF SISSINGH AND ZEEMAN.

18. A more decisive test of the present theory is obtained by comparing its results with the elaborate series of experiments which have been recently made by Drs. Sissingh, Zeeman, and Wind, at the laboratory of Leyden. These consist in observations of the amplitude ($\mu$) and phase ($m$) of the "magneto-optic component" of light reflected from magnetised mirrors of iron, nickel, and cobalt, for various angles of incidence. The details of the definition of these quantities will be found in Sissingh's paper in the *Archives Néerlandaises**; $\mu$ is always reckoned on the supposition that the amplitude of the incident light is unity, and $m$ is defined as retardation of phase calculated relatively to that component of the ordinary metallic reflexion which is polarised in the plane of incidence. The values of these quantities corresponding to the particular cases when the incident light is polarised in, or perpendicularly to the plane of incidence, are distinguished by the suffixes ($_i$) and ($_p$) respectively. It may readily be shewn that the components of the incident light in the directions defined by Sissingh as "principal directions," are in the present notation represented by $-A_0 \sec i$ and $-B_0$; while the corresponding principal components of the reflected light are $-A \sec i$ and $-B$.

19. When the incident light is polarised in the plane of incidence $B_0 = 0$; and in formulae (xxxiv) the incident ray is represented by $-A_0 \sec i$, the magneto-optic component of the reflected ray by $-B$, and the component relatively to which phase

* Sissingh, "Mesures relatives au phénomène de Kerr," *Archives Néerlandaises*, vol. XXVII.

is to be measured, by $-A \sec i$. Hence

$$360^\circ - m_i = \text{vector angle of } \{B \cos i/A\}_{B_0=0}$$

$$= \text{vector angle of } \frac{16\pi^2 C_0 e^{i\alpha} \iota V^{-1} \lambda^{-1} \mathfrak{M}^{-1} (\alpha_0 \sin^2 i + \gamma_0 \mathfrak{M}^2) \cos i}{(\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)} \quad \text{.....(xxxvii).}$$

When the incident light is polarised perpendicularly to the plane of incidence $A_0 = 0$, and the incident ray is represented by $-B_0$; the magneto-optic component of the reflected ray is $-A \sec i$, or rather that term of $-A \sec i$ that contains the factor $B_0$. The ray relatively to which phase is measured is represented by that term of $-A \sec i$ which contains the (vanishing) factor $A_0$. If $A_0$ be supposed to be only just not zero, then, since the incident ray is supposed to be plane polarised, $B_0/A_0$ is a real quantity. Hence we have

$$360^\circ - m_p = \text{vector angle of } \frac{16\pi^2 C_0 e^{i\alpha} \iota V^{-1} \lambda^{-1} (\cos^2 i/\mathfrak{M}) (\alpha_0 \sin^2 i - \gamma_0 \mathfrak{M}^2)}{(\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)} \quad \text{...(xxxviii).}$$

From (xxxvii) and (xxxviii) we see at once that when the reflexion is equatorial, that is when $\gamma_0 = 0$,

$$m_i = m_p = m \text{ (say)};$$

this agrees with the observation of Sissingh who, from the results of his experiments on equatorial reflexion, came to the conclusion that for any given angle of incidence the magneto-optic component has the same amplitude and phase, whether the incident light be polarised in or perpendicularly to the plane of incidence.

We also see that when the reflexion is polar, that is when $\alpha_0 = 0$,

$$m_i = m_p \pm 180^\circ.$$

Now Zeeman*, as a result of experiments on polar reflexion, came to the conclusion that for all angles of incidence $m_i = m_p$: here therefore is a discrepancy.

20. When the reflexion is equatorial, we see from (xxxvii) that

$$360^\circ - m = \text{vector angle of } \frac{16\pi^2 C_0 e^{i\alpha} \iota V^{-1} \lambda^{-1} \alpha_0 \sin^2 i \cos i}{\mathfrak{M} (\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)}.$$

In determining $m$ from this expression there is an ambiguity to the extent of 180°, for in defining $m$ Sissingh requires that it shall not be altered when $\alpha_0$ changes sign. Examining his paper, we see that his standard case corresponds to $\alpha_0$ negative. We shall also assume $C_0$ negative; in what follows the consequences of the alternative assumption, viz. $C_0$ positive, are obtained by adding 180° to the calculated values of $m$. We now find

$$m = 270^\circ - \alpha + \text{the sum of the vector angles of}$$
$$\mathfrak{M},\ (\mathfrak{M} - \cos i), \text{ and } (\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i) \text{.....(xxxix),}$$

and, to get $m$ accurately for any particular angle of incidence, these three vector angles must be calculated from formula (xxxii), using the known values of $R$ and $\alpha$ for the particular metal considered.

* Zeeman, "Mesures relatives au phénomène de Kerr," *Archives Néerlandaises*, vol. XXVII. p. 252.

The following table shews the results of Sissingh's observations on the phase for various angles of incidence, and the theoretical values of the phase for the same incidences, calculated from the present theory.

*Equatorial Reflexion from Iron. Yellow Light.* $a_0 = -1400$ C.G.S.

| Angle of incidence | Calculated value of $m$ | Sissingh's observed value of $m$ | Excess of $m$ (observed) over $m$ (calculated) |
|---|---|---|---|
| 86° 0′ | 199° 55′ $-x$ | 209° 26′ | 9° 31′ $+x$ |
| 82° 30′ | 192° 39′ $-x$ | 204° 22′ | 11° 43′ $+x$ |
| 76° 30′ | 183° 51′ $-x$ | 194° 49′ | 10° 58′ $+x$ |
| 71° 25′ | 178° 33′ $-x$ | 190° 3′ | 11° 30′ $+x$ |
| 61° 30′ | 171° 38′ $-x$ | 181° 49′ | 10° 11′ $+x$ |
| 51° 22′ | 167° 8′ $-x$ | 179° 0′ | 11° 52′ $+x$ |
| 36° 10′ | 162° 47′ $-x$ | 174° 9′ | 11° 22′ $+x$ |

In order that there should be agreement of theory with experiment it is necessary that the value of $x$ for iron should be about $-11°$; if this be so the agreement is extremely good.

21. When the reflexion is polar we see from (xxxviii) that

$$360° - m_p = \text{vector angle of } \frac{-16\pi^2 C_0 e^{\iota x} \iota V^{-1} \lambda^{-1} \gamma_0 \cos^2 i \mathfrak{M}^2}{(\mathfrak{M} - \cos i)(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)}.$$

Taking $\gamma_0$ positive in the standard case, and still assuming $C_0$ negative, we find that

$m_p = 270° - x -$ vector angle of $\mathfrak{M}^2$

$+$ the sum of the vector angles of $(\mathfrak{M} - \cos i)$ and $(\mathfrak{M} R^{-2} e^{-2\iota a} + \cos i)$.....(xl);

and $m_i$ differs from this by 180°.

Observations of the amplitude and phase of the magneto-optic component, in the case of polar reflexion from an iron mirror, have been made by Zeeman. An account of these experiments will be found in the paper which we have already referred to; he confines himself to one angle of incidence, viz. $i = 51° 22'$. His result as regards phase compares with theory as follows—

*Polar Reflexion from Iron. Yellow Light.* $\gamma_0 = +850$ C.G.S.

| Angle of incidence | Calculated value of $m_i$ | Zeeman's observed value of $m$ | Excess of $m$ (observed) over $m_i$ (calculated) |
|---|---|---|---|
| 51° 22′ | 151° 14′ $-x$ | 229° 55′ | 78° 41′ $+x$ |

Thus the agreement of $m_i$ with Zeeman's $m$ requires that $x = -78° 41'$. The same value of $x$ would correspond to agreement between $m_p$ and Zeeman's $m$ if $C_0$ were

assumed positive; both results are at variance with that of the preceding paragraph. Values of $x$ numerically greater than 90° are excluded by the definition in § 16, and so, of course, need not be discussed.

22. In considering the amplitude of the magneto-optic component, it is to be noticed that, when the incident light is polarised in the plane of incidence, the incident ray is $-A_0 \sec i$ and the magneto-optic component is $-B$; when the incident light is polarised perpendicularly to the plane of incidence, the incident ray is $-B_0$ and the magneto-optic component is $-A \sec i$. Hence

$$\mu_i = \text{mod}\left(\frac{B\cos i}{A_0}\right)_{B_0=0}, \quad \mu_p = \text{mod}\left(\frac{A}{B_0\cos i}\right)_{A_0=0} \quad \text{.....(xli).}$$

Thus, for equatorial reflexion, we readily derive from (xxxiv)

$$\mu_i = \text{mod}\,\frac{16\pi^2 C_0 e^{i\alpha} \iota V^{-1}\lambda^{-1}a_0 \sin^3 i\cos i}{\mathfrak{M}(\mathfrak{M}+\cos i)(\mathfrak{M}R^{-2}e^{-2\iota\alpha}+\cos i)},$$

$$\mu_p = \text{the same};$$

and therefore $\mu_i = \mu_p = \mu$ (say), which agrees with Sissingh's result.

If for brevity we put $16\pi^2 C_0 V^{-1}\lambda^{-1}a_0 \equiv L$, then

$$\mu = L\,.\,\text{mod}\,\frac{\sin^3 i\cos i}{\mathfrak{M}(\mathfrak{M}+\cos i)(\mathfrak{M}R^{-2}e^{-2\iota\alpha}+\cos i)},$$

and the latter factor may be calculated for any angle of incidence.

In the following table the values of $\mu$ derived from theory for various angles of incidence are compared with the values observed by Sissingh.

*Equatorial Reflexion from Iron. Yellow Light.* $a_0 = -1400$ C.G.S.

| Angle of incidence | Calculated value of $\log_{10}\mu - \log_{10}L$ | Sissingh's observed value of $10^3 \times \mu$ | $\left(\frac{\text{Calculated value of } \mu/L}{\text{Observed value of } \mu}\right)$ |
|---|---|---|---|
| 86° 0′ | $\bar{2}$·1485 | ·284 | 49·57 |
| 82° 30′ | $\bar{2}$·3438 | ·530 | 41·64 |
| 76° 30′ | $\bar{2}$·4673 | ·715 | 41·02 |
| 71° 25′ | $\bar{2}$·4932 | ·815 | 38·20 |
| 61° 30′ | $\bar{2}$·4512 | ·820 | 34·46 |
| 51° 22′ | $\bar{2}$·3228 | ·760 | 27·67 |
| 36° 10′ | $\bar{3}$·9724 | ·630 | 14·90 |
| 24° 16′ | $\bar{3}$·5853 | ·430 | 8·95 |
| 12° 0′ | $\bar{4}$·6192 | ·260 | 1·60 |
| 6° 0′ | $\bar{5}$·7233 | ·125 | ·423 |

In order that the theory should agree with experiment it is necessary that all the numbers in the last column should be equal. Obviously this is not the case; and their inequality is so pronounced, and depends in such a regular manner upon the angle of incidence, that it cannot possibly be attributed to accidental errors of observation. We must therefore conclude that here the theory is distinctly at variance with experiment.

23. For polar reflexion, we derive from (xli) and (xxxiv)

$$\mu_p = \text{mod}\,\frac{-16\pi^2 C_0 e^{ix_i} V^{-1}\lambda^{-1}\mathfrak{M}^2\gamma_0 \cos i}{(\mathfrak{M}+\cos i)(\mathfrak{M}R^{-2}e^{-2ia}+\cos i)}$$

$$= -\mu_i = \mu\,(\text{say}).$$

Comparing this with the amplitude in equatorial reflexion, we find

$$\frac{\mu\,(\text{equatorial})}{\mu\,(\text{polar})} = \text{mod}\left[\frac{(-\alpha_0)\sin^3 i}{\gamma_0\,\mathfrak{M}^3}\right].$$

If $-\alpha_0 = 1400$, $\gamma_0 = 850$, $i = 51°\,22'$, the value of this ratio for iron, as calculated from theory, is ·0122. But the values ascribed to $\alpha_0$, $\gamma_0$, and $i$ correspond to the experiments of Sissingh and Zeeman; and the latter found experimentally

$$\frac{\mu\,(\text{Sissingh})}{\mu\,(\text{Zeeman})} = \cdot 294,$$

so that here again there is a serious discrepancy between theory and experiment.

*Nickel.*

24. In the paper of Zeeman's already quoted there are given some measurements which he made upon polar reflexion from nickel. He also quotes experimental results of Kundt* and Drude†, which he expresses in a form similar to his own. These I have used to form the following tables, wherein the theoretical values of the phase and amplitude have in all cases been calculated for yellow light.

*Equatorial Reflexion from Nickel. White Light.*

| Angle of incidence | Calculated value of $m$ | Kundt's observed value of $m_i$ | Excess of $m_i$ (observed) over $m$ (calculated) |
|---|---|---|---|
| 30° 6′ | 146° 11′ − $x$ | 176° 10′ | 29° 59′ + $x$ |
| 40° 0′ | 148° 50′ − $x$ | 115° 42′ | − 33° 8′ + $x$ |
| 50° 0′ | 152° 13′ − $x$ | 115° 14′ | − 36° 59′ + $x$ |
| 61° 30′ | 158° 17′ − $x$ | 127° 39′ | − 30° 38′ + $x$ |
| 65° 18′ | 161° 1′ − $x$ | 126° 42′ | − 34° 19′ + $x$ |
| 75° 0′ | 170° 45′ − $x$ | 130° 6′ | − 40° 39′ + $x$ |

* *Wied. Ann.* vol. XXIII.

† *Wied. Ann.* vol. XLVI.

The value of $x$ indicated by the figures in the last column is about $+35°$; this would give fairly good agreement except in the case of the first angle of incidence.

*Equatorial Reflexion from Nickel. White Light.*

| Angle of incidence | Calculated value of $m$ | Drude's observed value of $m$ | Excess of $m$ (observed) over $m$ (calculated) |
|---|---|---|---|
| 60° | $157° 19' - x$ | 131° 38′ | $-25° 41' + x$ |
| 65° | $160° 48' - x$ | 133° 57′ | $-26° 51' + x$ |
| 75° | $170° 45' - x$ | 191° 41′ | $20° 56' + x$ |
| 80° | $178° 17' - x$ | 171° 18′ | $-6° 59' + x$ |

If the third angle of incidence, for which Drude's result differs widely from that of Kundt, be left out of account the mean value indicated for $x$ is about $+17°$.

*Equatorial Reflexion from Nickel.*

| Angle of incidence | Calculated value of $\log_{10}\mu - \log_{10}L$ | Kundt's observed value of $10^3 \times \mu$ | $\left(\frac{\text{Calculated value of } \mu/L}{\text{Observed value of } \mu}\right)$ |
|---|---|---|---|
| 30° 6′ | $\bar{3}$·8199 | ·21 | 31·46 |
| 40° 0′ | $\bar{2}$·1366 | ·77 | 17·79 |
| 50° 0′ | $\bar{2}$·3527 | 1·39 | 16·21 |
| 61° 30′ | $\bar{2}$·5017 | ·90 | 35·28 |
| 65° 18′ | $\bar{2}$·5278 | ·84 | 40·14 |
| 75° 0′ | $\bar{2}$·5334 | ·23 | 145·1 |

The inequality of the numbers in the last column shews that the theory does not here agree with experiment.

*Polar Reflexion from Nickel. Yellow Light.*

| Angle of incidence | Calculated value of $m_i$ | Zeeman's observed value of $m$ | Excess of $m$ (observed) over $m_i$ (calculated) |
|---|---|---|---|
| 50° | $160° 19' - x$ | 191° 40′ | $32° 21' + x$ |

so that the value of $x$ required for agreement is $-32° 21'$.

The experimental results used in the following table are due to Dr C. H. Wind*.

*Polar Reflexion from Nickel. Yellow Light.*

| Angle of incidence | Calculated value of $m_i$ | Observed value of $m_i$ | Excess of $m_i$ (observed) over $m_i$ (calculated) |
|---|---|---|---|
| 39° 4′ | 155° 35′ − $x$ | 14° 32′ | − 141° 3′ + $x$ |
| 55° 0′ | 163° 1′ − $x$ | 17° 47′ | − 145° 3′ + $x$ |
| 75° 0′ | 180° 30′ − $x$ | 32° 25′ | − 148° 5′ + $x$ |

As the value of $x$ here indicated, viz. about + 145°, is inadmissible, it appears that this set of experiments requires $C_0$ to be positive; the value of $x$ then indicated is − 35°.

*Cobalt.*

25. Experiments made by Zeeman and by Drude on mirrors of cobalt are used in the following tables.

*Polar Reflexion from Cobalt. White Light.*

| Angle of incidence | Calculated value of $m_i$ | Zeeman's observed value of $m$ | Excess of $m$ (observed) over $m_i$ (calculated) |
|---|---|---|---|
| 45° | 157° 55′ − $x$ | 200° 34′ | 42° 39′ + $x$ |
| 60° | 165° 6′ − $x$ | 207° 40′ | 42° 34′ + $x$ |
| 73° | 175° 54′ − $x$ | 217° 55′ | 42° 1′ + $x$ |

Here the value indicated for $x$ is − 42° approximately.

*Polar Reflexion from Cobalt.*

| Angle of incidence | Calculated value of $m_i$ | Zeeman's observed value of $m$ | Excess of $m$ (observed) over $m_i$ (calculated) |
|---|---|---|---|
| 50° | 159° 55′ − $x$ | 205° 9′ | 45° 14′ + $x$ |
| 60° | 165° 6′ − $x$ | 212° 30′ | 47° 24′ + $x$ |
| 72° | 174° 47′ − $x$ | 225° 51′ | 51° 4′ + $x$ |

The indicated value of $x$ is about − 47°.

* *Communications from the Leiden Laboratory of Physics*, No. 9.

*Equatorial Reflexion from Cobalt.*

| Angle of incidence | Calculated value of $m$ | Drude's observed value of $m$ | Excess of $m$ (observed) over $m$ (calculated) |
|---|---|---|---|
| 35° | 146° 49′ − $x$ | 102° 36′ | − 44° 13′ + $x$ |
| 60° | 155° 48′ − $x$ | 154° 33′ | − 1° 15′ + $x$ |
| 75° | 168° 29′ − $x$ | 167° 4′ | − 1° 25′ + $x$ |
| 83° | 182° 19′ − $x$ | 167° 3′ | − 15° 16′ + $x$ |

No value of $x$ will make the theory agree with this series of experiments; the mean of the indicated values is about + 15° 30′.

The experiments used in the following table are described in the *Communications from the Leiden Laboratory of Physics*, No. 5.

*Polar Reflexion from Cobalt. White Light.* $\gamma_0 = 430$ C.G.S.

| Angle of incidence | Calculated value of $\log_{10}\mu_p - \log_{10} L'$ | Zeeman's observed value of $10^2 \times \mu$ | $\left(\frac{\text{Calculated value of } \mu_p/L'}{\text{Observed value of } \mu}\right)$ |
|---|---|---|---|
| 45° | ·5559 | 1·58 | 2276 |
| 60° | ·5349 | 1·50 | 2284 |
| 73° | ·4690 | 1·17 | 2516 |

In this, $L'$ is an abbreviation for $-16\pi^2 C_0 V^{-1}\lambda^{-1}\gamma_0$. The approximate equality of the numbers in the last column indicates a very good agreement of the theory with this set of experiments.

## Conclusion.

26. On comparing with one another the results of the last six paragraphs it is readily seen that, while it is possible to assign to the $x$ of any one of the metals considered such a value as will bring the theory into a more or less rough agreement with the experiments on equatorial reflexion, or again such a value as will bring about agreement with the experiments on polar reflexion, yet these two values of $x$ are so widely separated from one another that they cannot be reconciled even by the utmost allowances for errors of observation. The results as regards amplitude moreover, in the cases of nickel and iron, shew that no two of the experiments can be accounted for

by the same value of $C_0$. The discrepancies between the theory and the Kerr experiments are also very noticeable, though their importance is perhaps not so great on account of the extreme delicacy which is required in these experiments. On the whole then, it is clear that the theory which we have been considering does not account for the observed facts. A confirmation of this conclusion is afforded by the absence of $\beta_0$ from the formulae (xxxiv), which signifies that, according to the theory, the component of magnetisation perpendicular to the plane of incidence produces no effect; but such an effect does exist, and has been measured by Zeeman*.

* *Communications from the Leiden Laboratory of Physics*, No. 29.

# III. *On the solutions of the equation* $(V^2+\kappa^2)\psi=0$ *in elliptic coordinates and their physical applications.* By R. C. MACLAURIN, St John's College.

[*Received* and *read* 16 May, 1898.]

IT is well known that the solution of a very large number of physical problems depends almost entirely on the successful treatment of the differential equation $(V^2+\kappa^2)\psi=0$. The difficulty in any case is to obtain a solution in terms of coordinates that lend themselves readily to the symbolic expression of the "boundary conditions" of the problem. When the boundaries are either right circular cylinders or spheres all the analytical difficulties have been most successfully overcome, but comparatively little headway has been made with other forms of bounding surfaces.

The present paper deals with problems relating to elliptic cylinders and spheroids.

The two-dimensional problem seems first to have been attacked by Mathieu [Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique, *Journal de Liouville*, t. XIII., p. 137]. This was in 1868. In the following year H. Weber published a paper in the *Math. Annalen* (Bd. I.) dealing with the subject. Further references will be found in Heine [*Handbuch der Kugelfunctionen*, Bd. II., p. 208] and in a recent work by Pockels (1891) *Ueber die partielle Differentialgleichung* $\Delta u+\kappa' u=0$.

The three-dimensional problem is from an analytical point of view very similar to the one for two dimensions. It has been attacked by C. Niven in the *Phil. Trans.* 1880, in a paper on the "Conduction of Heat in ellipsoids."

The present essay will be found to contain very little in common with any of the above—except that the physical problem that occupied Prof. Niven in 1880 receives a brief mention here, although the method of treatment is quite different. Since this paper was written, my attention has been called to a short article by Lindemann, "Ueber die Differentialgleichung der Functionen des Elliptischen Cylinders" [*Math. Annalen*, Bd. 22, p. 117]. He uses independent variables practically the same as those of this essay (p. 43, et seq.) and obtains some of the results reached here, but is mainly occupied with proving some theorems about the *product* of two solutions of the differential equation.

In dealing with elliptic cylinders, we may define the position of any point by its distance $z$ measured along the axis from some fixed normal section and by the semi-axes $a$ and $a'$ of the confocal ellipse and hyperbola that pass through the point. We may develope $\psi$ (regarded as a function of $z$) in a Fourier series of the form $\Sigma A_n \cos(nz-\epsilon_n)$, where the coefficients $A_n$ are functions of $a$ and $a'$. Since

$$\frac{\partial^2}{\partial z^2} A_n \cos(nz-t_n) = -n^2 A_n \cos(nz-\epsilon_n),$$

we see that the equation $(V^2+\kappa^2)\psi=0$ reduces to $(V_1^2+\kappa'^2)\psi=0$, where $\kappa'^2=\kappa^2-n^2$ and $V_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. Thus practically the whole difficulty is reduced to finding a suitable solution of the equation $(V_1^2+\kappa^2)\psi=0$.

Now with the usual notation $ds^2=\frac{da^2}{h_1^2}+\frac{da'^2}{h_2^2}$,

we have
$$V_1^2\psi = h_1h_2\left[\frac{\partial}{\partial a}\left(\frac{h_1}{h_2}\frac{\partial\psi}{\partial a}\right)+\frac{\partial}{\partial a'}\left(\frac{h_2}{h_1}\frac{\partial\psi}{\partial a'}\right)\right],$$

and if we take $x=a/h$; $x'=a'/h$ where $2h$ is the distance between the foci of the confocal system we get

$$V_1^2\psi = \frac{1}{h^2(x^2-x'^2)}\left[(x^2-1)\frac{\partial^2\psi}{\partial x^2}+x\frac{\partial\psi}{\partial x}-\left\{(x'^2-1)\frac{\partial^2\psi}{\partial x'^2}+x'\frac{\partial\psi}{\partial x'}\right\}\right].$$

If then $(V_1^2+\kappa^2)\psi=0$, we have, putting $h\kappa=\lambda$,

$$\lambda^2(x^2-x'^2)\psi = -\left[(x^2-1)\frac{\partial^2\psi}{\partial x^2}+x\frac{\partial\psi}{\partial x}\right]+(x'^2-1)\frac{\partial^2\psi}{\partial x'^2}+x'\frac{\partial\psi}{\partial x'}.$$

$$(x^2-1)\frac{\partial^2\psi}{\partial x^2}+x\frac{\partial\psi}{\partial x}+\lambda^2x^2\psi = (x'^2-1)\frac{\partial^2\psi}{\partial x'^2}+x'\frac{\partial\psi}{\partial x'}+\lambda^2x'^2\psi.$$

Now put $\psi=yy'$ where $y$ is a function of $x$ only and $y'$ of $x'$, and we get

$$\frac{1}{y}\left[(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\lambda^2x^2y\right] = \frac{1}{y'}\left[(x'^2-1)\frac{d^2y'}{dx'^2}+x'\frac{dy'}{dx'}+\lambda^2x'^2y'\right]$$
$$= p^2 \text{ say, where } p^2 \text{ is some constant.}$$

Hence we have
$$(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-p^2)y=0,$$

and a similar equation for $y'$ in terms of $x'$.

We have $x=\frac{a}{h}=\frac{1}{e}$. Thus $x$ is the reciprocal of the eccentricity of the ellipse and so is always *greater* than unity.

Also $x'=a'/h=\frac{1}{e'}$, and $x'$, being the reciprocal of the eccentricity of the hyperbola, is always *less* than unity.

We see then that everything now depends on the solution of the equation

$$(x^2-1)\,y''+xy'+(\lambda^2x^2-p^2)\,y=0.$$

This equation has three critical points, $x=+1$, $x=-1$, $x=\infty$.

Hence we must endeavour to obtain suitable solutions for the three domains corresponding to these critical points.

To obtain a solution in the neighbourhood of $x=1$ we make the substitution $x=-z+1$ in the above equation, which becomes

$$z\,(z-2)\,y''+(-z+1)\,y'+[\lambda^2\,(z-1)^2-p^2]\,y=0.$$

If we write this in the normal form (i.e. with the coefficient of $y''$ unity) we see at once, by Fuch's Theorem, that its integrals are *regular* in the neighbourhood of $z=0$.

Hence $y$ is of the form:—

$$y=a_0z^m+a_1z^{m+1}+\ldots\ldots+a_nz^{m+n}+\ldots\ldots,$$

there being at the most only a *finite* number of negative powers of $z$.

The *indicial equation* proves to be $m(2m-1)=0$, so that we have two series corresponding to $m=0$ and $m=\frac{1}{2}$.

Equating the coefficient of $z^{m+n}$ to zero we get:—

$$-(m+n+1)(2m+2n+1)\,a_{n+1}+(\overline{m+n}^2+\lambda^2-p^2)\,a_n-2\lambda^2a_{n-1}+\lambda^2a_{n-2}=0.$$

Thus for the series corresponding to $m=0$, we have

$$-(n+1)(2n+1)\,a_{n+1}+(n^2+\lambda^2-p^2)\,a_n-2\lambda^2a_{n-1}+\lambda^2a_{n-2}=0\ldots\ldots\ldots\ldots(1),$$

and for that corresponding to $m=\frac{1}{2}$,

$$-(n+1)(2n+3)\,a_{n+1}+(\overline{n+\tfrac{1}{2}}^2+\lambda^2-p^2)\,a_n-2\lambda^2a_{n-1}+\lambda^2a_{n-2}=0\ldots\ldots\ldots(2).$$

Now consider the first series ($m=0$) and put $v_{n+1}=a_{n+1}/a_n$.

Then we have $$v_{n+1}=+\frac{n^2+\lambda^2-p^2}{(n+1)(2n+1)}-\frac{2\lambda^2}{(n+1)(2n+1)v_n}+\frac{\lambda^2}{(n+1)(2n+1)v_nv_{n-1}}$$

$$=+\frac{1}{2}+\frac{An+B}{(n+1)(2n+1)}-\frac{2\lambda^2}{(n+1)(2n+1)v_n}+\frac{\lambda^2}{(n+1)(2n+1)v_nv_{n-1}}.$$

Thus when $n$ *is very large, either* $v_n$ is indefinitely small *or* $v_n$ approaches the limit $+\frac{1}{2}$.

The series is therefore convergent if $|z|<2$.

It is easy to show that the series also converges if $|z|=2$.

For this purpose, put $z=+2z_1$, then we have

$$\phi(z)=\sum_0^\infty a_n z^n=\sum_0^\infty b_n z_1^n=\phi_1(z_1) \text{ say, where } b_n=(+2)^n a_n,$$

$$(n+1)(2n+1)b_{n+1}=2(n^2+\lambda^2-p^2)b_n-8\lambda^2(b_{n-1}-b_{n-2}),$$

$$b_{n+1}=\frac{2(n^2+\lambda^2-p^2)}{(n+1)(2n+1)}b_n-\frac{8\lambda^2}{(2n+1)(n+1)}(b_{n-1}-b_{n-2}).$$

Thus when $n$ is very great $b_{n+1}=b_n$, so that $\dfrac{b_{n-1}-b_{n-2}}{(2n+1)(n+1)}$ is negligible and we may write $b_{n+1}=\dfrac{2(n^2+\lambda^2-p)}{(n+1)(2n+1)}b_n$.

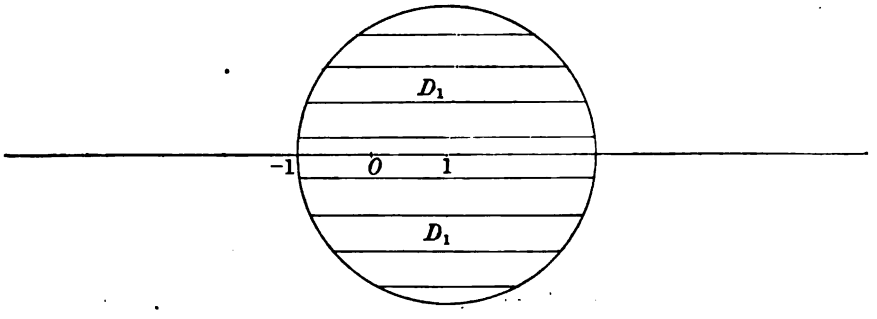
$\underset{n=\infty}{\text{Lt}}\, n\left(\dfrac{b_n}{b_{n-1}}-1\right)=\dfrac{3}{2}$, and as this is greater than unity it follows that the series $\sum_0^\infty b_n=\phi_1(1)$ is *convergent;* so that $\phi(+2)$ is convergent.

We can prove in exactly the same way that the other series (corresponding to $m=\frac{1}{2}$) is convergent if $|z|\not>2$.

We have thus obtained two solutions of our differential equation appropriate to the neighbourhood of the critical point $x=1$. These are:—

$$P=\phi(z)=\sum_0^\infty a_n z^n \; [a_n \text{ given by (1) p. 43, } a_0=1],$$

$$Q=z^{\frac{1}{2}}\psi(z)=\sum_0^\infty a_n z^{n+\frac{1}{2}} \; [a_n \text{ ..... (2) ..... } a_0=1].$$




The '*domain*' of these functions $P$ and $Q$ is the interior of the circle, whose centre is $z=0$ and radius $=2$; i.e. the circle with centre at the critical point $x=1$, and passing through the next critical point $x=-1$. We shall call this the domain $D_1$.

$P = \phi(z) = \phi(1-x)$ is a 'uniform' function, returning to its original value, when the argument $z$ traces out any closed contour. On the other hand

$$Q = z^{\frac{1}{2}}\psi(z) = (1-x)^{\frac{1}{2}}\psi(1-x)$$

is 'multiform'. $\psi(z)$ is uniform, but $z^{\frac{1}{2}}$ changes sign if $z$ makes a tour round the pole $z=0$. If $\rho$ is the modulus and $\theta$ the amplitude of $z$ we may take

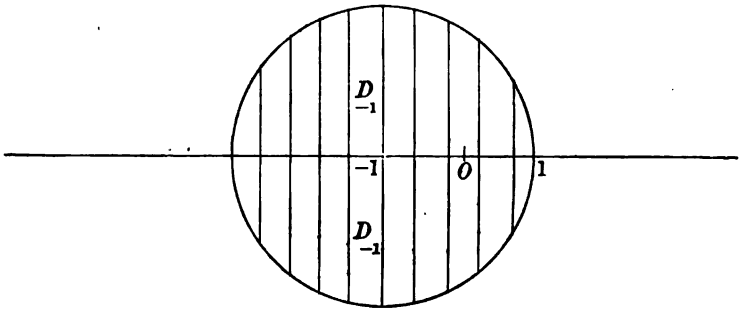
$$z^{\frac{1}{2}} = +\sqrt{\rho}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$$

Our fundamental equation is not changed if for $x$ we write $-x$. Hence following out the same argument as above we shall get two solutions in the neighbourhood of the critical point $x=-1$, viz.:—

$$P' = \phi(1+x), \quad Q' = (1+x)^{\frac{1}{2}}\psi(1+x),$$

where $\phi$ and $\psi$ are the functions already obtained.

The domain of these functions $P'$ and $Q'$ is the interior of the circle with centre at the critical point $x=-1$ and passing through the next critical point $x=1$. We shall refer to this as the domain $D_{-1}$.




We must now turn to the consideration of the integrals in the neighbourhood of the third critical point $x=\infty$. For this purpose we make the substitution $x = \frac{1}{x_1}$, a substitution which is simply and elegantly represented in a geometrical form by the aid of Neumann's sphere—in the well-known manner.

Our equation now becomes:—

$$x_1^4(1-x_1^2)y'' + x_1^3(1-2x_1^2)y' + (\lambda^2 - p^2x_1^2)y = 0.$$

The critical points of this equation are

$$x_1 = 0, \quad x_1 = \pm 1,$$

corresponding to

$$x = \infty, \quad x = \pm 1.$$

We have to consider the solutions in the neighbourhood of $x_1 = 0$.

Writing this equation in the normal form

$$y''+p_1y'+p_2y=0,$$

we see that $x_1=0$ is a pole of $p_1$ of order 1 and of $p_2$ of order 4. It follows from Fuch's theorem that the *integrals in the neighbourhood of* $x_1=0$ *are irregular.* We must not then expect quite the same simplicity in the treatment here as that which characterised the earlier work.

From the form of our equation we see that if $y$ be expanded in powers of $x_1$ the coefficients of even and odd powers will be quite independent so that we may assume two solutions in the forms:—

$$\begin{aligned} y &= x_1^s(a_0+a_1x_1^2 \quad +\ldots+a_nx_1^{2n} \quad +\ldots \\ &\qquad\qquad +a_{-1}/x_1^2+\ldots+a_{-n}/x_1^{2n}+\ldots) \\ &= x_1^s\sum_{n=-\infty}^{n=\infty} a_nx_1^{2n}, \end{aligned}$$

and

$$y=x_1^{s'}\sum_{n=-\infty}^{n=\infty} a_n'x_1^{2n+1}.$$

Take the first series $y=x_1^s\sum_{-\infty}^{\infty} a_nx_1^{2n}$ and substitute in the equation

$$x_1^4(1-x_1^2)y''+x_1^3(1-2x_1^2)y'+(\lambda^2-p^2x_1^2)y=0.$$

We must have:—

$$\begin{aligned} &x_1^4(1-x_1^2)\left[\frac{s(s-1)a_0}{x_1^2}+(s+2)(s+1)a_1+(s+4)(s+3)a_2x_1^2+\ldots\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.+(s+2n)(s+2n-1)a_nx_1^{2n-2}+\ldots\right] \\ &\quad\left[+\frac{(2-s)(3-s)a_{-1}}{x_1^4}+\frac{(4-s)(5-s)a_{-2}}{x_1^6}+\ldots+\frac{(2n-s)(2n+1-s)a_{-n}}{x_1^{2n+2}}+\ldots\right] \\ &\quad+x_1^3(1-2x_1^2)\left[\frac{sa_0}{x_1^2}+(s+2)a_1x_1+(s+4)a_2x_1^3+\ldots+(s+2n)a_nx_1^{2n-1}+\ldots\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left.-\frac{(2-s)a_{-1}}{x_1^3}-\frac{(4-s)a_2}{x_1^5}-\ldots-\frac{(2n-s)a_{-2n}}{x_1^{2n+1}}\right] \\ &\quad+(\lambda^2-p^2x_1^2)\left[a_0+a_1x_1^2+\ldots+a_nx_1^{2n}+\ldots\right. \\ &\qquad\qquad\qquad \left.+\frac{a_{-1}}{x_1^2}+\ldots+\frac{a_{-n}}{x_1^{2n}}+\ldots\right]\equiv 0. \end{aligned}$$

Hence we must have:

$$\begin{aligned} \lambda^2a_0 &+[(2-s)^2-p^2]\,a_{-1}-(3-s)(4-s)\,a_{-2} &&=0, \\ \lambda^2a_1 &+[s^2-p^2]\,a_0 \qquad -(2-s)(1-s)\,a_{-1} &&=0, \\ \lambda^2a_{n+1} &+[\overline{2n+s^2}-p^2]\,a_n \quad -(2n-1+s)(2n-2+s)\,a_{n-1} &&=0. \end{aligned}$$

The last equation holding also if for $n$ we write $-n$.

Considering the ascending part of the series (for which $n$ is +) we see at once from the last relation that in the general case ($s$ unrestricted) the series $a_n$, $a_{n+1}\ldots$ will continually increase with $n$, so that the series formed in the way indicated on the last page will usually be divergent and so useless. But by properly choosing $s$ it may be possible to make $a_n = 0$ for indefinitely large values of $n$, in which case as we shall prove our series will converge in the region $|x_1| \not> 1$. It is easy to see that if the ascending part of the series converges, the descending part will also converge.

For we have $(2n+1-s)(2n+2-s)\,a_{-(n+1)} - (\overline{2n-s}^2 - p^2)\,a_{-n} - \lambda^2 a_{-(n-1)} = 0.$

Let $a_{-n} = \dfrac{\lambda^{2n}(-1)^n c_n}{\Pi(2n-s)}$ where $\Pi(x)$ is Gauss' function $= \Gamma(x+1)$ and we get

$$\lambda^2 c_{n+1} + [\overline{2n-s}^2 - p^2]\,c_n - (2n-s)(2n-s-1)\,c_{n-1} = 0.$$

Comparing this with the relation between $a_{n+1}$, $a_n$ and $a_{n-1}$, we see that for very large values of $n$, the relations are practically identical. It is easy to see that $c_n$ cannot be infinite for any *finite* value of $n$, hence it follows from what has just been said that for *large values of* $n$ we may put $a_{-n} = \dfrac{\kappa\lambda^{2n}(-1)^n a_n}{\Pi(2n-s)}$ where $\kappa$ is finite.

If then the ascending part of the series converges, the descending part will do so with great rapidity for *any finite value* of the argument.

We have said that $c_n$ cannot be infinite for any *finite* value of $n$. For, putting

$$p^2 - \overline{2n-s}^2 = -\lambda^2 v_n,$$

$$(2n-s)(2n-s-1) = -\lambda^2 u_{n-1},$$

we have $c_{+1} + v_n c_n + u_{n-1} c_{n-1} = 0$. Suppose we make

$$c_1 = 0,$$

$$c_0 = 1,$$

then we have a system of equation to determine $c_2$, $c_3 \ldots c_n \ldots$

$$c_2 = -u_0,$$

$$c_3 + v_2 c_2 = 0,$$

$$c_4 + v_3 c_3 + u_2 c_2 = 0,$$

$$c_5 + v_4 c_4 + u_3 c_3 \qquad = 0,$$

and so on.

Solving we get

$$c_2 = -u_0,$$

$$c_3 = -\begin{vmatrix} -u_0 & 1 \\ 0 & v_2 \end{vmatrix} = u_0 v_2,$$

$$c_4 = \begin{vmatrix} -u_0 & 0 & 1 \\ 0 & 1 & u_2 \\ 0 & v_2 & u_3 \end{vmatrix},$$

and so on; the determinant in the denominator of $c_n$ being $= (-1)^n$.

Thus as the denominators cannot vanish and the numerators cannot become infinite for finite values of $n$, we conclude that $c_n$ is necessarily finite when $n$ is so.

Returning now to p. 47, we have seen that $s$ must be chosen so as to make $a_\infty = 0$, if the series is to converge.

The condition $a_\infty = 0$ is of course a necessary, but not a sufficient, condition for convergence. But it is easy to show that when this relation is satisfied the series *does* converge in the region

$$|x_1| < 1, \quad \{\text{i.e. } |x| > 1\}.$$

The series we are considering is

$$\Sigma u_n = x_1^s \Sigma a_n x_1^{2n},$$

where $$\lambda^2 a_{n+1} + [(2n+s)^2 - p^2] a_n - (2n+s-1)(2n+s-2) a_{n+1} = 0.$$

For large values of $n$ we have $a_{n+1}$ very small, by hypothesis, and then the above relation gives

$$\text{Lt}\, \frac{a_n}{a_{n-1}} = \text{Lt}\, \frac{(2n+s-1)(2n+s-2)}{(2n+s)^2 - p^2} = 1;$$

$$\therefore \ \text{Lt}\, \frac{u_n}{u_{n-1}} = x_1^2 \ \text{[we are considering the } \textit{ascending} \text{ powers of the series].}$$

Hence the ascending series is convergent if $|x_1| < 1$, and divergent if $|x_1| > 1$.

When $|x_1| = 1$, we have $\text{Lt}\, \frac{u_n}{u_{n-1}} = 1$, so that the higher test $\text{Lt}\, n \left[\frac{u_n}{u_{n+1}} - 1\right]$ must be used. This limit is $\frac{3}{2}$, so that the series converges *on* the circle $|x_1| = 1$.

Thus we have proved that the *ascending* part of the series converges when

$$|x_1| \not> 1, \ \text{i.e. } |x| \not< 1.$$

But by p. 47 if $a_n$ is finite the *descending* part converges for *all* finite values of the argument. Of course when the ascending part converges $a_n$ is finite, so that the *whole* series converges in any region in which the ascending part is convergent. Clearly also, if the ascending part *diverges* the series as a whole is divergent.

Summing up then, we find the necessary and sufficient condition for the convergence of the series we have obtained as a formal solution of our differential equation in the region $|x_1| \not> 1$ is that $s$ should be a root of the equation $a_\infty = 0$.

For brevity, put

$$(2n+s)^2 - p^2 = \lambda^2 v_n; \ (2n+s-1)(2n+s-2) = -\lambda^2 u_{n-1},$$

and we have $$a_{n+1} + u_n a_n + u_{n+1} a_{n-1} = 0.$$

Some of the constants are necessarily arbitrary. Taking $a_0 = 1$, $a_1 = 0$, we have

$$a_1 = -v_0,$$

$$a_2 + v_1 a_1 = -u_0,$$

$$a_3 + v_2 a_2 + u_1 a_1 = 0, \quad \text{and so on.}$$

Solving, we get

$$a_1 = -v_0;\ a_2 = \begin{vmatrix} v_0 & 1 \\ v_0 & v_1 \end{vmatrix};\ a_3 = -\begin{vmatrix} v_0 & 0 & 1 \\ u_0 & 1 & v_1 \\ 0 & v_2 & u_1 \end{vmatrix} = \begin{vmatrix} v_0 & 1 & 0 \\ u_0 & v_1 & 1 \\ 0 & u_1 & v_2 \end{vmatrix};$$

$$a_\infty = \begin{vmatrix} v_0 & 1 & 0 & 0 & 0 & 0 & 0 \ldots \\ u_0 & v_1 & 1 & 0 & 0 & 0 & 0 \ldots \\ 0 & u_1 & v_2 & 1 & 0 & 0 & 0 \ldots \\ 0 & 0 & u_2 & v_3 & 1 & 0 & 0 \ldots \\ 0 & 0 & 0 & u_3 & v_4 & 1 & 0 \ldots \\ 0 & 0 & 0 & 0 & u_4 & v_5 & 1 \ldots \\ 0 & \ldots\ldots\ldots\ldots & & & & & \\ \vdots & \ldots\ldots\ldots\ldots\ldots\ldots & & & & & \end{vmatrix},$$

and $s$ is determined as a root of $a_\infty = 0$.

We proceed to show how $s$ may be developed in a series of ascending powers of $\lambda^2$.

Let $$a'_n = \lambda^{2n} a_n, \quad v_n = p^2 - (2n+s)^2,$$

then we have $$a'_{n+1} = v_n a'_n + \lambda^2 (s+2n-1)(s+2n-2)\, a'_{n-1};\ a'_1 = v_0;$$

$$a_2 = v_0 v_1 + \lambda^2 s(s+1) = v_0 v_1 \left[1 + \lambda^2 . \frac{s.s+1}{v_0 v_1}\right];$$

$$a'_3 = v_0 v_1 v_2 \left[1 + \lambda^2 . \left(\frac{s.s+1}{v_0 . v_1} + \frac{s+2.s+3}{v_1 . v_2}\right)\right],$$

and so on. The equation determining $s$ is $a'_\infty = 0$.

It is clear that this equation is equivalent to

$$v_0 v_1 v_2 \ldots v_\infty [1 + \lambda_0^2 S_1 + \lambda_0^4 S_2 + \lambda_0^6 S_2 \ldots] = 0,$$

where ${}_0S_1 = \dfrac{s.s+1}{v_0 . v_1} + \dfrac{(s+2)(s+3)}{v_1 . v_2} + \dfrac{s+4.s+5}{v_2 . v_3} \ldots$ (mode of formation is obvious),

${}_0S_2 =$ sum of products of every *two* non-adjacent terms of the last series,

${}_0S_3 =$ sum of products of every *three* non-adjacent terms of that series; and so on.

If $\lambda = 0$ we have $v_n = 0$ where $n$ is zero or any positive integer, and this gives

$$s = \pm p - 2n.$$

To indicate the method of procedure let us obtain a few terms of the expansion of $s$, corresponding to $v_0 = 0$, as a first approximation.

We have $${}_0S_1 = \frac{s.s+1}{v_0 . v_1} + {}_1S_1,$$

$${}_0S_2 = \frac{s.s+1}{v_0 . v_1}\, {}_2S_1 + {}_1S_2;\quad {}_1S_2 = \frac{s+2.s+3}{v_1 v_2}\, {}_3S_1 + {}_2S_2,$$

$${}_0S_3 = \frac{s.s+1}{v_0 . v_1}\, {}_2S_2 + {}_1S_3;\quad {}_1S_3 = \frac{s+2.s+3}{v_1 v_2}\, {}_3S_2 + {}_2S_3,$$

and so on.

Our equation to determine $s$ is $v_0[1+\lambda^2{}_0S_1+\lambda^4{}_0S_2+\ldots\ldots]=0$,

i.e. $$v_0+\frac{s\,.\,s+1}{v_1}[\lambda^2+\lambda^4{}_2S_1+\lambda^6{}_2S_2+\ldots\ldots]+v_0\lambda^2[{}_1S_1+\lambda^2{}_1S_2+\ldots\ldots]=0.$$

*First approximation:*

$$v_0=0,\quad p^2-s^2=0,\quad s=p \text{ (taking + sign).}$$

*Second:*

$$v_0+\lambda^2.\frac{s\,.\,s+1}{v_1}=0,$$

$$p^2-s^2-\lambda^2.\frac{p(p+1)}{4(p+1)}=0;\quad s=p-\frac{\lambda^2}{8}.$$

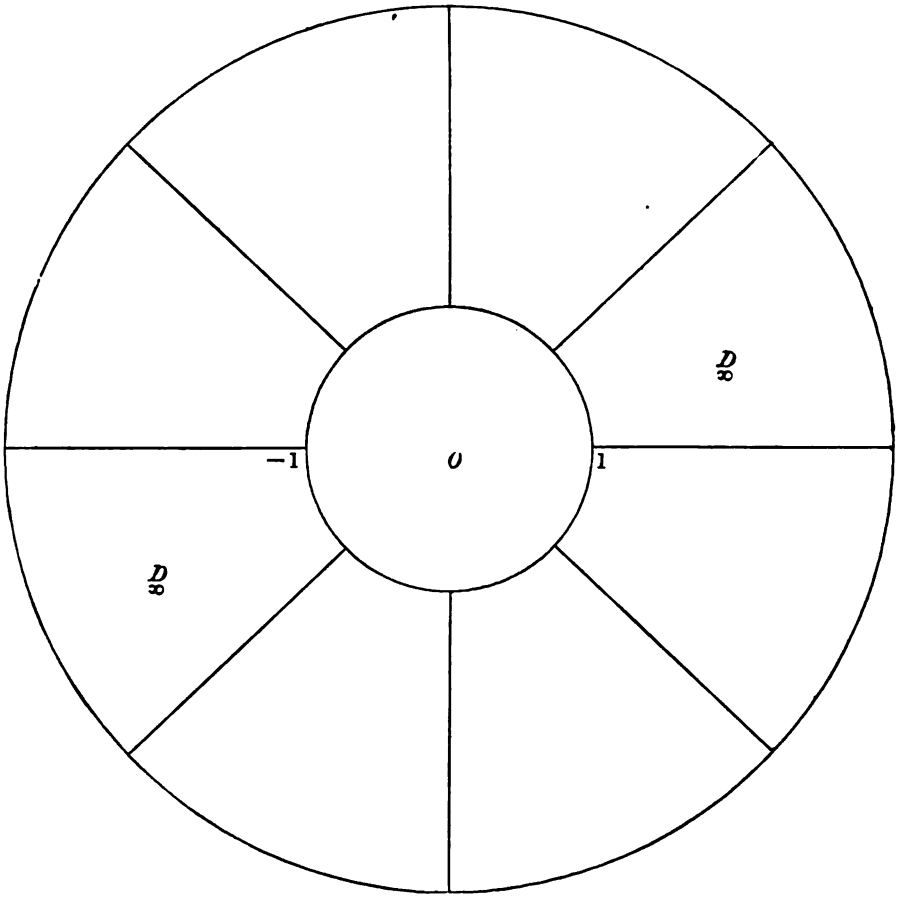
*Third:*

$$v_0+\frac{s(s+1)}{v_1}\lambda^2+\frac{s(s+1)}{v_1}\lambda^4{}_2S_1-\frac{s\,.\,s+1}{v_1}\lambda^4{}_1S_1=0,$$

$$v_0+\frac{s(s+1)}{v_1}.\lambda^2-\frac{s\,.\,s+1\,.\,s+2\,.\,s+3}{v_1{}^2v_2}\lambda^4=0,$$

$$s^2=p^2-\frac{p\lambda^2}{4}+\frac{4+7p-p^2}{128\,.\,(p+1)}\lambda^4,$$

and so on.



We have now shown how to obtain six different solutions of our fundamental equation:—

$$P = \phi(1-x); \text{ and } Q = (1-x)^{\frac{1}{2}}\psi(1-x),$$

which are applicable to the domain $D_1$,

$$P' = \phi(1+x); \text{ and } Q' = (1+x)^{\frac{1}{2}}\psi(1+x) \ldots\ldots D_{-1},$$

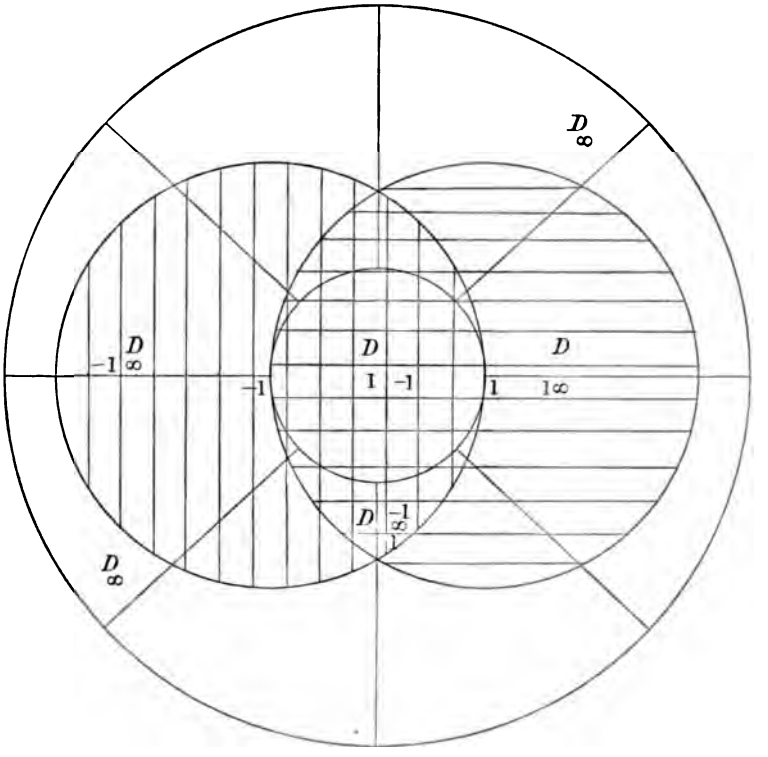
$$P'' = x_1^{s}\Sigma a_n x_1^{2n} = x^{-s}\Sigma a_n x^{-2n} = x^{-s}\Phi(x),$$

and

$$Q'' = x_1^{s'}\Sigma a_n' x_1^{2n+1} = x^{-s'}\Sigma a_n' x^{-2n-1} = x^{-s'}\Phi(x).$$

The last two series are convergent for all *finite* values of $x$ such that $|x| > 1$. Thus the domain $D_\infty$ is the ring bounded by the two circles $|x| = 1$, and $|x| = \infty$.

If we draw these various domains we see that they overlap.




The region common to two domains such as $D_1$ and $D_{-1}$ will be referred to as the domain $D_{1-1}$.

In the domain $D_{1-1}$ we have four solutions $P$, $Q$, $P'$, $Q'$, and there must consequently be a linear relation between any three of these.

We proceed to determine the values of the various constants in these linear relations.

In the domain $D_{1-1}$ we have $P'=AP+BQ$,

i.e. $$\phi(1+x)=A\phi(1-x)+B(1-x)^{\frac{1}{2}}\psi(1-x).$$

Make $x$ approach the point $x=1$ and we get $\phi(2)=A\phi(0)=A$.

[We have proved, p. 44, that $\phi(2)$ is finite.]

Next, keeping $x$ real let it approach the point $x=-1$, so that the amplitude of $1-x$ is $\pi$, then we get

$$1=\phi(0)=A\phi(2)+Bi\sqrt{2}\,\psi(2).$$

Thus we have $A=\phi(2)$; $B=\dfrac{i}{\sqrt{2}}\cdot\dfrac{-1+\phi^2(2)}{\psi(2)}$.

Also we have in the same domain $Q'=A'P+B'Q$,

i.e. $$(1+x)^{\frac{1}{2}}\psi(1+x)=A'\phi(1-x)+B'(1-x)^{\frac{1}{2}}\psi(1-x).$$

Making $x=1$ we get $\sqrt{2}\,\psi(2)=A'\phi(0)=A'$.

And making $x$ move along the real axis to $x=-1$,

we get $$0=A'\phi(2)+B'i\sqrt{2}\,\psi(2).$$

Hence we have $$A'=\sqrt{2}\,\psi(2);\quad B'=+i\phi(2).$$

We may note that $\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}=-i$.

We have then, $\begin{matrix} P'=AP+BQ \\ Q'=A'P+B'Q \end{matrix}$, from which we deduce

$$P=\frac{\begin{vmatrix} P' & B \\ Q' & B' \end{vmatrix}}{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}}=i(+BQ'-B'P'),$$

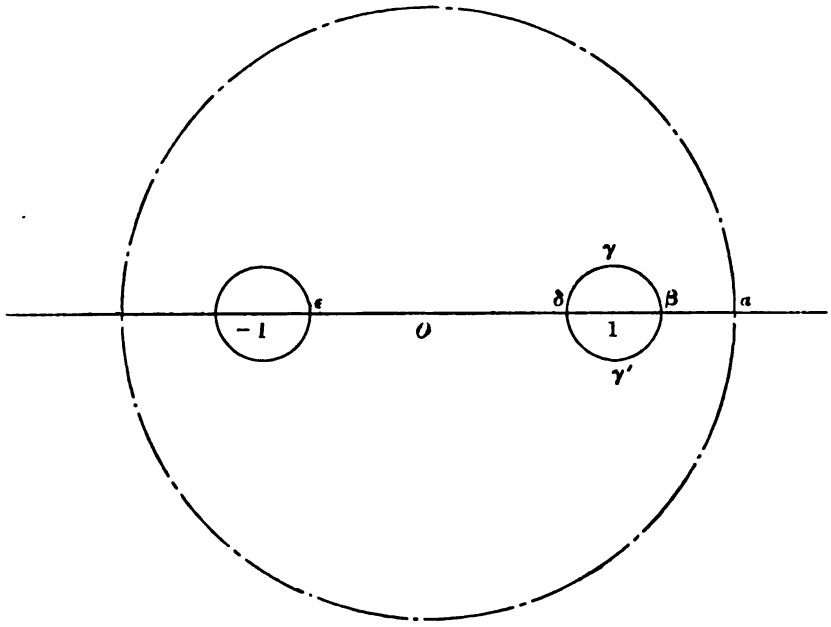
and $$Q=i(-AQ'+A'P').$$

These relations having been found we can determine with what solution of the differential equation we shall reach any point $\beta$ (say) in $D_1$ or $D_{-1}$ when we start with a definite solution from a point $\alpha$ in $D_1$ or $D_{-1}$ and move along a given path to $\beta$, provided the path avoids the critical points 1, and $-1$.

As an example of this we shall work out a particular case that will afterwards be of service. Suppose we start at $\alpha$ on the real axis in the region $D_1$ and move along the dotted path in the figure so as to come back to $\alpha$.

The only poles of the functions involved are the points 1 and $-1$, so that the effect of going along the dotted road is the same as going along the path $\alpha\beta\gamma\delta\epsilon$, round the loop enclosing $-1$, then back along $\epsilon\delta\gamma\beta\alpha$, then along $\alpha\beta\gamma\delta\gamma'\alpha$.

Suppose then we start at $\alpha$ with the solution $P$ which is appropriate to $D_1$. As soon as we get to $\delta$ we can (if we choose) express $P$ in the form $i(BQ' - B'P')$ for




we are now in the domain $D_{1-1}$. The functions $Q'P'$ hold throughout the domain $D_{-1}$ and so are suitable for use when we wish to make a tour round $-1$. On making the circuit round $-1$, $P'$ will be unaffected, while $Q'$ will change sign. Thus we get back to $\epsilon$ with the solution $-i(BQ' + B'P')$. We are again in the domain $D_{1-1}$ so that we can express $P'$ and $Q'$ in terms of $P$ and $Q$ by the help of the relations just obtained. We have, in fact,

$$\begin{aligned} -i(BQ' + B'P') &= -i[B(A'P + B'Q) + B'(AP + BQ)] \\ &= -i[(AB' + A'B)P + 2BB'Q]. \end{aligned}$$

$P$ and $Q$ hold all along the path $\epsilon\delta\gamma\beta\alpha$, so that we arrive at $\alpha$ with the solution

$$-i[(AB' + A'B)P + 2BB'.Q].$$

Starting now with this solution and going along $\alpha\beta\gamma\delta\gamma'\beta\alpha$ we note that $P$ and $Q$ hold all along the path, and that on making the circuit round 1, $P$ is unaltered while $Q$ changes sign. We conclude then that if we start from $\alpha$ with the solution $P$ and go along the dotted path we shall return to $\alpha$ with the solution

$$-i[(AB' + A'B)P - 2BB'Q].$$

In exactly the same way we might show that if we had started with the solution $Q$ we should have returned with $-i[(AB' + A'B)Q - 2AA'.P]$.

Next consider the various solutions that we have in the domain $D_{1\infty}$. They are:—

$$\left.\begin{aligned} P &= \phi(1-x);\ Q = (1-x)^{\frac{1}{2}}\psi(1-x) \\ P'' &= x^{-\nu}\Phi(x);\ Q'' = x^{-\nu'}\Psi(x) \end{aligned}\right\}.$$

We must therefore have relations of the form

$$x^{-s}\Phi(x)=P''=\alpha P+\beta Q=\alpha\phi(1-x)+\beta(1-x)^{\frac{1}{2}}\psi(1-x),$$
$$x^{-s'}\Psi(x)=Q''=\alpha' P+\beta' Q=\alpha'\phi(1-x)+\beta'(1-x)^{\frac{1}{2}}\psi(1-x).$$

The work on the last page will enable us to get a relation between $\alpha$ and $\beta$. For suppose we start with the function $P''$ from a point such as $a$ on p. 53 in the domain $D_{1\infty}$ and describe a contour enclosing the points 1 and $-1$ but entirely confined to the domain $D_\infty$ so that $P''$ holds throughout this path. We shall return to $a$ with the function $(\cos 2s\pi - i\sin 2s\pi)P''$.

But in the region $D_{1\infty}$, $P''=\alpha P+\beta Q$, and from page 53 we see that if we start with the function $\alpha P+\beta Q$ and describe a path such as the one we have just followed with $P''$ we return to the starting point with the function

$$-i\alpha[(AB'+A'B)P-2BB'.Q]-i\beta[(AB'+A'B)Q-2AA'.P].$$

This then must be identical with $(\cos 2s\pi - i\sin 2s\pi)(\alpha P+\beta Q)$. Equating the coefficients of $P$ and $Q$ in these identities we get

$$\alpha(\cos 2s\pi - i\sin 2s\pi)=-i\alpha(AB'+A'B)+i.2AA'.\beta,$$
$$\beta(\cos 2s\pi - i\sin 2s\pi)=-i\beta(AB'+A'B)+i.2BB'.\alpha.$$

These two equations are really identical, they give us

$$\frac{\beta}{\alpha}=\frac{2iBB'}{\cos 2s\pi - i\sin 2s\pi + i(AB'+A'B)}.$$

Now we have $x^{-s}\Phi(x)=\alpha\phi(1-x)+\beta(1-x)^{\frac{1}{2}}\psi(x)$.

Putting $x=1$ we get $\Phi(1)=\alpha\phi(0)=\alpha$, so that we have

$$\alpha=\Phi(1);\ \beta=\frac{2iBB'.\Phi(1)}{\cos 2s\pi - i\sin 2s\pi + i(AB'+A'B)}.$$

Similarly we can determine $\alpha'$ and $\beta'$.

We have now completed the formal solution of our differential equation. It is an equation of the second order with three critical points, and we have obtained two solutions in the domain of each critical point and determined the constants in the linear relations that connect different solutions in a common domain. There is no finite region of the plane for which we have not obtained an appropriate solution. But for dealing with physical problems which it is the main object of this paper to attack, the solutions in the vicinity of the origin are not in a very convenient form. In the domain $|x|<1$ we want solutions expressed in powers of $x$. It is easy to build up such solutions by taking proper linear functions of $P.Q.P'.Q'$, but we may as well attack the problem directly.

Our equation is $(x^2-1)y''+xy'+(\lambda^2x^2-p^2)y=0$. Assuming a solution of the form $y=a_0x^m+a_1x^{m-1}+\ldots$ we find, on equating the coefficient of the lowest power of $x$ to zero, that we must have $m(m-1)=0$.

Thus we have two solutions of the forms:—

$$y = a_0 + a_1 x^2 + \ldots + a_n x^{2n} + \ldots$$
$$y = c_0 x + c_1 x^3 + \ldots + c_n x^{2n+1} + \ldots,$$

The equations connecting the coefficients are:—

$$\begin{cases} -2a_1 - p^2 a_0 = 0\,; \; -12a_2 + (4 - p^2)\, a_1 + \lambda^2 a_0 = 0 \\ -(2n+1)(2n+2)\, a_{n+1} + (4n^2 - p^2)\, a_n + \lambda^2 a_{n-1} = 0 \end{cases},$$

$$\begin{cases} -6c_1 + (1 - p^2)\, c_0 = 0\,; \; -20c_2 + (3^2 - p^2)\, c_1 + \lambda^2 c_0 = 0 \\ -(2n+2)(2n+3)\, c_{n+1} + [(2n+1)^2 - p^2]\, c_n + \lambda^2 c_{n-1} = 0 \end{cases}.$$

[We may note here that if $\lambda = 0$ our equation reduces to $(x^2 - 1)\, y'' + xy' - p^2 y = 0$, which is satisfied by $y = A \cos p\theta + B \sin p\theta$ where $x = \cos\theta$; $|x| \not> 1$. For some purposes it is convenient to take $z = \lambda x$ as variable and then we have $(z^2 - \lambda^2)\, y'' + zy' + (z^2 - p^2)\, y = 0$, which, when $\lambda = 0$, reduces to Bessel's equation and gives $y = AJ_p(z) + BK_p(z)$.]

Let us examine the convergence of these series.

We have $\quad -(2n+1)(2n+2)\, a_{n+1} + (4n^2 - p^2)\, a_n + \lambda^2 a_{n-1} = 0.$

Let $v_n = a_{n+1}/a_n$ and we get $v_n = \dfrac{4n^2 - p^2}{(2n+1)(2n+2)} + \dfrac{\lambda^2}{(2n+1)(2n+2)} \cdot \dfrac{1}{v_{n-1}}$

$$= 1 - \frac{6n + 2 + p^2}{(2n+1)(2n+2)} + \frac{\lambda^2}{(2n+1)(2+2)\, v_{n-1}}.$$

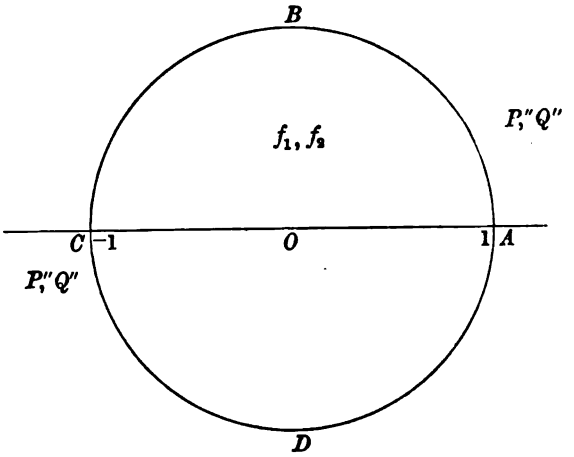
Thus *either* $v_n$ is very small when $n$ is large, in which case $v_n$ approaches the limit $\dfrac{-\lambda^2}{(2n+1)(2n+2)}$, *or* $v_n$ is not indefinitely small and approaches the limit 1. In the former case the series converges for *all* finite values of the argument. In the latter it converges when $|x| < 1$, and also, as we can easily show by proceeding as on p. 44, when $|x| = 1$. Thus in the most unfavourable case the series converges when $|x| \not> 1$ and this quite independently of the value of $p$. Exactly similar reasoning applies to the odd series $c_0 x + \ldots$. Denoting these solutions by $f(x)$ [two functions, one odd, and the other even] we have four solutions of our differential equation expressed as series of powers of $x$. Two solutions ($f$) are confined to the region $|x| < 1$, the other two ($P''$ and $Q''$) to the region $|x| \not< 1$. All these series hold *on* the circle $|x| = 1$.

If we start from $A$ (see next page) and go with $f_1$ or $f_2$ along the real axis to $C$ and back again to $A$, we must return to $A$ with the *same* function with which we started if the functions involved are suitable for use in *physical problems* dealing with the *complete* cylinder.

For a tour round the cylinder must bring us back to the *same* physical conditions from which we started.

Now *on* the circle $|x| = 1$ we can express $f_1$ and $f_2$ as linear functions of $P''$ and $Q''$. And starting with $f$ from $A$ and going along $AOCOA$ must bring us to the same

result as going along $ABCDA$. But along this latter path we can replace $f$ by a linear function of $P''$ and $Q''$; hence it follows that, in dealing with physical problems concerning the *complete* cylinder, $P''$ and $Q''$ must be such that they return to their




original values when taken round the contour $ABCDA$. It is at once obvious that this requires *that $s$ and $s'$ should be either zero,* or *positive integers* (we can take them to be zero).

But if $s$ and $s'$ are zero, the condition for the convergence of the series $P''$ and $Q''$ will restrict us to a *particular set of values of $p$,* viz. the roots of $a_\infty=0$, [p. 48].

Now a few pages back, in dealing with the convergence of the $f$ functions, we saw that there were *two* alternatives—$v_n$ tends either to zero or unity in the limit. In the latter case the series is convergent in *a certain region* $f$ whatever be the value of $p$. In the former the series is convergent for *all* finite values of the argument. If then we choose $p$ as a root of $v_\infty=0$, we shall have series that are convergent for all portions of the $x$ plane at a finite distance from the origin.

Now if $s=0$ the relations connecting the coefficients on p. 46, are the same as those that connect the coefficients of the $f$ functions.

We see then that in this case the particular values of $p$ to which we are confined for the convergence of our functions $P''$ and $Q''$ are identical with the roots of $v_\infty=0$. [To distinguish the odd and even series we shall refer to these as the roots of $a_\infty=0$ and $c_\infty=0$ respectively.] Thus the $f$ functions are convergent for *all finite* values of the argument and our problem is to a certain extent simplified by the fact that we can use the same functions [an odd and even power series of $x$, respectively] for *all* finite values of $x$. However, this simplification is counterbalanced by the fact that the two solutions on p. 55, i.e. the odd and even series correspond to different values of $p^2$, so that we are obliged to complete the solution of our differential equation by the aid of new functions.

As the determination of the appropriate values of $p^2$ is important for the physical applications, we must consider this part of the problem in some detail and obtain some numerical results.

Considering the even series (the discussion of the odd series proceeds on exactly the same lines), we have seen that $p^2$ must be chosen so as to be a root of $a_\infty = 0$.

Putting $a_0 = 1$, the various coefficients are given by the equations

$$2a_1 = -p^2,$$

$$12a_2 = (4 - p^2)\, a_1 + \lambda^2,$$

$$(2n+1)(2n+2)\, a_{n+1} = (4n^2 - p^2)\, a_n + \lambda^2 a_{n-1}.$$

Before proceeding with the actual calculation of the roots of $a_\infty = 0$, we shall notice some points as to the position of the various roots of $a_n = 0$.

The equation $a_n = 0$ considered as an equation in $p^2$ is clearly of the $n$th degree. Its roots are all positive, for it is obvious that when $p^2$ is negative $a_n$ is necessarily positive. For if this is true of $a_n$ and $a_{n-1}$, then since

$$(2n+1)(2n+2)\, \lambda^2 a_{n+1} = (4n^2 - p^2)\, a_n + \lambda^2 a_{n-1},$$

it is true also of $a_{n+1}$. But it is clearly true of $a_1$ and $a_2$ so that, by induction, it must be true for $a_n$.

For some purposes it is rather more convenient to replace $a_n$ by $(-1)^n a_n'$, so that we have:—

$$2a_1' = p^2;\ \ 12a_2' = (p^2 - 4)\, a_1' + \lambda^2,$$

$$(2n+1)(2n+2)\, a'_{n+1} = (p^2 - 4n^2)\, a_n' + \lambda^2 a'_{n-1}.$$

It is now obvious that all the roots of $a'_{n+1} = 0$ are less than $4n^2$. For if we put $p^2 = 4n^2$ or any greater quantity $a_1' a_2' \ldots\ldots a'_{n+1}$ are all positive.

Again when $p^2 = (2n-2)^2$, $a_1' a_2' \ldots\ldots a_n'$ are all positive and

$$a'_{n+1} = \frac{-8n+4}{(2n+1)(2n+2)}\, a_n' + \frac{\lambda^2 a'_{n-1}}{(2n+1)(2n+2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

For large values of $n$ the last term on the right is negligible compared with the first, and as $a_n'$ is positive it follows that for large values of $n$, $a'_{n+1}$ will be negative. [Our interest is centred mainly in the roots of $a_n = 0$ *when $n$ is large*, and in what follows we shall suppose $n$ large enough to make the right-hand side of (1) negative when $a_n'$ is positive.]

We have seen then that $a'_{n+1}$ is positive when $p^2 = (2n)^2$, and negative when

$$p^2 = (2n-2)^2.$$

Hence the equation $a'_{n+1} = 0$ has a root between $(2n-2)^2$ and $(2n)^2$.

We have seen that all the roots of $a_n'=0$ are less than $(2n-2)^2$. Let $p_n^2$ be any such root. Then the expressions

$$p_n^2-(2n)^2;\quad p_n^2-(2n+2)^2;\ldots\ldots$$

are all *negative*.

Also since $p^2=p_n^2$ satisfies $a_n'=0$ we have these relations:—

$$(2n+3)(2n+4)\,a'_{n+2}=[p_n-(2n+2)^2]\,a'_{n+1},$$

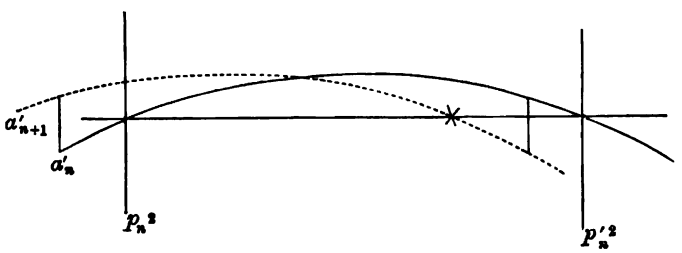
$$(2n+5)(2n+6)\,a'_{n+3}=[p_n-(2n+4)^2]\,a'_{n+2}+\lambda^2 a'_{n+1}.$$

etc. etc.

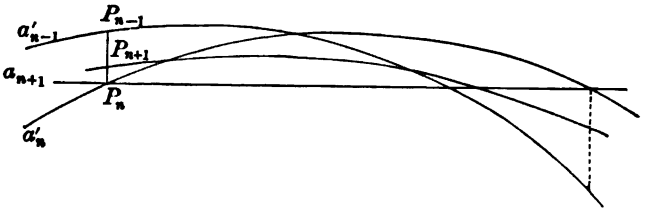
Thus when $p^2=p_n^2$ we see that $a'_{n+2}$ is of opposite sign to $a'_{n+1}$; and $a'_{n+3}$ is of opposite sign to $a'_{n+2}$ and so on [$n$ being large enough—see remark on last page]. If $a'_{n-1}$ is positive then $a'_{n+1}$ is positive, $a'_{n+2}$ is negative, and so on.

Now let $p_n'^2$ be the *next* root of $a_n'=0$ greater than $p_n^2$.

Suppose the graph of $a_n'$ is as in the figure, the dotted line representing $a'_{n+1}$. We have taken $a_n'$ to be negative when $p^2$ is a little less than $p_n^2$. For this value




of $p^2$, $a'_{n+1}$ must be of opposite sign to $a_n'$ and so positive. Next consider a value of $p^2$ slightly less than $p_n'^2$. Here $a_n'$ is positive and $a'_{n+1}$ consequently negative. Thus $a'_{n+1}$ has changed sign in the interval. Hence we see that a root of $a'_{n+1}=0$ lies between each pair of roots of $a_n'=0$. From what was said above this is true also of $a'_{n+2}=0$, $a'_{n+3}+0$; etc. Each of these equations has a root lying between any pair of roots of $a_n'=0$.



From the relation $(2n+1)(2n+2)\,a'_{n+1} = (p^2 - 4n^2)\,a_n' + \lambda^2 a'_{n-1}$ we see that when $a_n' = 0$, $a'_{n+1}$ and $a'_{n-1}$ have the same signs and that for large values of $n$, $a'_{n+1}$ is then but a small fraction of $a'_{n-1}$.

Thus if in the figure $P_nP_{n-1}$ is finite, $P_nP_{n+1}$ is *very* small (for large values of $n$)—so that the roots of $a'_{n+1} = 0$ are very close to those of $a_n' = 0$.

We conclude then that as $n$ increases the roots of $a'_{n+1} = 0$, approximate more and more closely to the roots of $a_n' = 0$ and so in the limit when $n = \infty$, the roots of $a_\infty = 0$ are *definite in position* and independent of $n$.

We shall now proceed to calculate some of the roots of $a_\infty = 0$ and $c_\infty = 0$. As on p. 49 we can express these quantities $a_\infty$ and $c_\infty$ as infinite determinants. But the second method there referred to is the practical one, i.e. we develope $p^2$ in a series of powers of $\lambda^2$, the series being rapidly convergent if $\lambda^2$ is not too large.

For brevity let $v_n = 4n^2 - p^2$; $a_n = \dfrac{u_n}{\underline{|2n}}$, then we get

$$u_{n+1} = u_n v_n + \lambda^2 . (2n-1)\,2n . u_{n-1}.$$

Taking $u_0 = 1$ we have

$$u_1 = v_0,$$

$$u_2 = v_0 v_1 \left[1 + \lambda^2 . \frac{1.2}{v_0 . v_1}\right],$$

$$u_3 = v_0 v_1 v_2 \left[1 + \lambda^2 \left(\frac{1.2}{v_0 . v_1} + \frac{3.4}{v_1 . v_2}\right)\right],$$

$$u_4 = v_0 v_1 v_2 v_3 \left[1 + \lambda^2 \left(\frac{1.2}{v_0 . v_1} + \frac{3.4}{v_1 . v_2} + \frac{5.6}{v_2 . v_3}\right) + \lambda^4 . \frac{1.2.5.6}{v_0 v_1 v_2 v_3}\right],$$

and so on.

We see then that the equation $a_\infty = 0$ is equivalent to

$$v_0 v_1 v_2 \ldots v_\infty \left[1 + \lambda_0^2 S_1 + \lambda_0^4 S_2 + \lambda_0^6 S_3 + \ldots\right] = 0,$$

where
$$_0S_1 = \frac{1.2}{v_0 . v_1} + \frac{3.4}{v_1 . v_2} + \frac{5.6}{v_2 . v_3} + \ldots,$$

$_0S_2$ = sum of products of every *two* non-adjacent terms of the last series, and so on; as on p. 49.

If $\lambda = 0$ we have $v_0 v_1 \ldots v_\infty = 0$, giving $p^2 = (2n)^2$, where $n$ is zero or any integer and leading to Bessel's functions $[J_0 J_1 \ldots]$.

We shall denote the values of $p^2$ that correspond to roots of $v_0 = 0$, $v_1 = 0$, etc., by $p_0^2$, $p_2^2$, etc., respectively.

$p_0^2$ (*corresponding to* $v_0=0$).

We have ${}_0S_1=\dfrac{1.2}{v_0.v_1}+\dfrac{3.4}{v_1.v_2}+\ldots$; ${}_1S_1=\dfrac{3.4}{v_1.v_2}+\ldots$ (the same as ${}_0S_1$ but beginning with second term) and similar notation for ${}_0S_2$, etc.

Clearly ${}_0S_1=\dfrac{1.2}{v_0.v_1}+{}_1S_1$; ${}_0S_2=\dfrac{1.2}{v_0.v_1}{}_2S_1+{}_1S_2$; ${}_0S_3=\dfrac{1.2}{v_0.v_1}{}_2S_2+{}_1S_3$ and so on.

$${}_1S_1=\frac{3.4}{v_1.v_2}{}_3S_1+{}_3S_2;\quad {}_1S_3=\frac{3.4}{v_1.v_2}{}_3S_3+{}_3S_3 \text{ and so on.}$$

So that our equation $v_0[1+\lambda^2{}_0S_1+\ldots]=0$ may be written

$$v_0+\frac{2}{v_1}(\lambda^2+\lambda^4{}_2S_1+\lambda^6{}_2S_2+\ldots)+v_0\lambda^2({}_1S_1+\lambda^2{}_1S_2+\ldots)=0.$$

*First approximation:* $\qquad v_0=0,\quad \therefore\ p_0^2=0.$

*Second.* $\qquad v_0+\dfrac{2}{v_1}\lambda^2=0.\quad \therefore\ v_0=-\dfrac{2\lambda^2}{v_1}=-\dfrac{\lambda^2}{2};\ p_0^2=\dfrac{\lambda^2}{2}.$

*Third.*

$$v_0+\frac{2\lambda^2}{v_1}+\frac{2\lambda^4}{v_1}.{}_2S_1-\frac{2\lambda^4}{v_1}.{}_1S_1=0,$$

$$v_0+\frac{2\lambda^2}{v_1}-\frac{2.3.4}{v_1^2v_2}\lambda^4=0,$$

$$\therefore\ 0=-p_0^2+\frac{2\lambda^2}{4-\frac{\lambda^2}{2}}-\frac{24}{16.16}\lambda^4=-p_0^2+\frac{\lambda^2}{2}-\frac{1}{3^2}\lambda^4\ldots$$

$$\therefore\ p_0^2=\frac{\lambda^2}{2}-\frac{\lambda^4}{32}.$$

*Fourth.*

$$\begin{aligned}0&=v_0+\frac{2\lambda^2}{v_1}+\frac{2\lambda^4}{v_1}{}_2S_1+\frac{2\lambda^6}{v_1}{}_2S_2+\left(-\frac{2\lambda^4}{v_1}+\frac{2.3.4}{v_1^2v_2}\lambda^6\right)({}_1S_1+\lambda^2{}_1S_2)\\
&=v_0+\frac{2\lambda^2}{v_1}+\frac{2\lambda^4}{v_1}({}_2S_1-{}_1S_1)+\frac{2\lambda^6}{v_1}\left({}_2S_2-{}_1S_1+\frac{3.4}{v_1v_2}{}_1S_1\right)\\
&=v_0+\frac{2\lambda^2}{v_1}-\frac{2\lambda^4}{v_1}.\frac{3.4}{v_1v_2}+\frac{2\lambda^6}{v_1}.\frac{3.4}{v_1v_2}\left(\frac{3.4}{v_1v_2}+\frac{5.6}{v_2v_3}\right)\\
&=-p_0^2+\frac{2\lambda^2}{4-\frac{\lambda^2}{2}+\frac{\lambda^4}{16}}-\frac{2\lambda^4.12}{\left(4-\frac{\lambda^2}{2}\right)^2\left(16-\frac{\lambda^2}{2}\right)}+\frac{24\lambda^6}{16.16}\left(\frac{12}{4.16}+\frac{30}{16.36}\right)\\
&=-p_0^2+\frac{\lambda^2}{2}-\frac{\lambda^4}{32}\ \text{(coefficient of } \lambda^6 \text{ vanish).}\end{aligned}$$

Thus up to $\lambda^6$ the expansion of $p_0^2$ in terms of $\lambda$ is

$$p_0^2=\frac{\lambda^2}{2}-\frac{\lambda^4}{32}.$$

$p_2^2$ (*corresponding to root* $v_1 = 0$).

Our equation is $\quad v_1[1 + \lambda^2 {}_0S_1 + \lambda^4 {}_0S_2 + \lambda^6 {}_0S_3 + \ldots] = 0. \qquad (1).$

$$ {}_0S_1 = \frac{1.2}{v_0 . v} + \frac{3.4}{v_1 . v_2} + {}_2S_1, $$

$$ {}_0S_2 = \frac{1.2}{v_0 . v_1} {}_2S_1 + \frac{3.4}{v_1 . v_2} . {}_3S_1 + {}_2S_2, $$

$$ {}_0S_0 = \frac{1.2}{v_0 . v_1} . {}_2S_2 + \frac{3.4}{v_1 . v_2} . {}_3S_2 + {}_2S_3. $$

(1) is equivalent to $v_1 + \frac{1.2}{v_0} . \lambda^2 (1 + \lambda^2 {}_2S_1 + \lambda^4 {}_2S_2 + \ldots) + \frac{3.4}{v_2} . \lambda^2 (1 + \lambda^2 {}_3S_1 + \lambda^4 {}_3S_2)$

$$ + v_1 \lambda^2 ({}_2S_1 + \lambda^2 {}_2S_1 + \lambda^4 {}_2S_3 + \ldots) = 0. $$

*First approximation:* $\quad v_1 = 0, \quad \therefore \; p_2^2 = 2^2 = 4.$

*Second.* $\quad v_1 + \lambda^2 \left( \frac{1.2}{v_0} + \frac{3.4}{v_2} \right) = 0, \quad \therefore \; v_1 = -\frac{\lambda^2}{2}; \; p_2^2 = 4 + \frac{\lambda^2}{2}.$

*Third.*

$$ \begin{aligned} 0 &= v_1 + \lambda^2 \left( \frac{1.2}{v_0} + \frac{3.4}{v_2} \right) + \lambda^4 \left( \frac{1.2}{v_0} . {}_2S_1 + \frac{3.4}{v_2} . {}_3S_1 - \frac{1.2}{v_0} . {}_2S_1 - \frac{3.4}{v_0} {}_3S_1 \right) \\ &= v_1 + \lambda^2 \left( \frac{1.2}{v_0} + \frac{3.4}{v_2} \right) + \lambda^4 \left[ \frac{3.4}{v_2} . ({}_3S_1 - {}_3S_1) \right] \\ &= v_1 + \lambda^2 \left( \frac{1.2}{v_0} + \frac{3.4}{v_2} \right) - \lambda^4 . \frac{3.4}{v_2} . \frac{5.6}{v_2 v_3}. \end{aligned} $$

Proceeding as before this gives $p_2^2 = 4 + \frac{\lambda^2}{2} + \frac{5\lambda^4}{12 \times 16}.$

*Fourth.*

$$ \begin{aligned} 0 &= v_1 + \lambda^2 \left( \frac{1.2}{v_0} + \frac{3.4}{v_2} \right) - \lambda^4 . \frac{3.4.5.6}{v_2^2 v_3} + \lambda^6 . \frac{3.4.5.6}{v_2^2 v_3} \left( \frac{5.6}{v_2 v_3} + \frac{7.8}{v_0 v_4} \right) \\ &= 4 - p_2^2 + \lambda^2 \left( \frac{-2}{4 + \frac{\lambda^2}{2} + \frac{5\lambda^4}{16.12}} + \frac{12}{12 - \frac{\lambda^2}{2} - \frac{5\lambda^4}{16.12}} \right) \\ &\quad - \lambda^4 . \frac{3.4.5.6}{\left(12 - \frac{\lambda^2}{2}\right)^2 \left(32 - \frac{\lambda^2}{2}\right)} + \lambda^6 . \frac{3.4.5.6}{12^2 . 32} \left( \frac{30}{12.32} + \frac{5.6}{32.60} \right). \end{aligned} $$

Here again the coefficient of $\lambda^6$ vanishes and we have up to $\lambda^8$,

$$ p_2^2 = 4 + \frac{\lambda^2}{2} + \frac{5\lambda^4}{12 \times 16}. $$

We shall turn next to the odd series. We have the general relation

$$ (2n+2)(2n+3)\, c_{n+1} = [(2n+1)^2 - p^2]\, c_n + \lambda^2 c_{n-1}. $$

In this case put $\quad v_n = (2n+1)^2 - p^2; \quad c_n = \frac{u_n}{2n+1},$

and we have $\quad u_{n+1} = v_n u_n + \lambda^2 . (2n+1)(2n)\, u_{n-1}.$

Taking $u_0=1$ we get $\quad u_1=v_0,$

$$u_2=v_0v_1\left(1+\lambda^2.\frac{2.3}{v_0.v_1}\right),$$

$$u_3=v_0v_1v_2\left[1+\lambda^2\left(\frac{2.3}{v_0.v_1}+\frac{4.5}{v_1.v_2}\right)\right],$$

and so on.

Just as before the equation to determine $p^2$ is

$$v_0v_1v_2\ldots v_\infty\,[1+\lambda^2{}_0S_1+\lambda^4{}_0S_2+\ldots]=0,$$

where
$$_0S_1=\frac{2.3}{v_0.v_1}+\frac{4.5}{v_1.v_2}+\frac{6.7}{v_2.v_3}+\ldots$$

[Everything is exactly as before except that

2.3 is associated with $v_0v_1$ instead of 1.2,

4.5 ..... $v_1v_2$ ..... 3.4,

and so on.]

$p_1^2$ (*corresponding to root* $v_0=0$).

*First approximation:* $v_0=0$; $\therefore\ p_1^2=1$.

*Second.*
$$v_0+\lambda^2.\frac{2.3}{v_1}=0;\ \therefore\ v_0=-\frac{3}{4}\lambda^2;\ p_1^2=1+\frac{3}{4}\lambda^2.$$

*Third.*
$$v_0=-\lambda^2.\frac{2.3}{v_1}+\lambda^4.\frac{2.3.4.5}{v_1^2v_2}$$

$$=-\frac{6\lambda^2}{8-\frac{3}{4}\lambda^2}+\frac{2.3.4.5\lambda^4}{8^2.24}$$

$$=-\frac{3}{4}\lambda^2+\frac{\lambda^4}{128};\ \therefore\ p_1^2=1+\frac{3}{4}\lambda^2-\frac{\lambda^4}{128}.$$

*Fourth.*
$$0=v_0+\frac{2.3}{v_1}\lambda^2-\frac{2.3.4.5}{v_1^2v_2}\lambda^4+\frac{2.3.4.5}{v_1v_2}\lambda^6\left(\frac{4.5}{v_1v_2}+\frac{6.7}{v_2v_3}\right)$$

$$=1-p_1^2+\frac{2.3\lambda^2}{8-\frac{3}{4}\lambda^2+\frac{\lambda^4}{128}}-\frac{2.3.4.5\lambda^4}{\left(8-\frac{3}{4}\lambda^2\right)^2\left(24-\frac{3}{4}\lambda^2\right)}$$

$$+\frac{2.3.4.5\lambda^6}{8.24}\left(\frac{4.5}{8.24}+\frac{6.7}{24.48}\right);$$

$$\therefore\ p_1^2=1+\frac{3}{4}\lambda^2-\frac{1}{128}\lambda^4-\frac{241}{12\times(32)^2}\lambda^6+\ldots.$$

$p_3^2$ (*root corresponding to* $v_1 = 0$).

Our equation is $$v_1[1+\lambda^2{}_0S_1+\lambda^4{}_0S_2+\ldots]=0 \quad\ldots\ldots\ldots\ldots(1),$$

and $$_0S_1=\frac{2.3}{v_0.v_1}+\frac{4.5}{v_1.v_2}+\frac{6.7}{v_2.v}+\ldots=\frac{2.3}{v_0.v}+\frac{4.5}{v_1v_2}+{}_2S_1,$$

$$_0S_2=\frac{2.3}{v_0v_1}{}_2S_1+\frac{4.5}{v_0.v_2}{}_2S_1+{}_2S_2;\quad {}_0S_3=\frac{2.3}{v_0v_1}{}_2S_2+\frac{4.5}{v_1.v_2}{}_2S_2+{}_2S_3.$$

Hence (1) becomes $$0=v_1+\frac{2.3}{v_0}\lambda^2[1+\lambda^2{}_2S_1+\lambda^4{}_2S_2+\ldots]$$
$$+\frac{4.5}{v_2}\lambda^2[1+\lambda^2{}_3S_1+\lambda^4{}_3S_2+\ldots]+v_1\lambda^2({}_2S_1+\lambda^2{}_2S_2+\ldots).$$

*First approximation:* $v_1=0$; $\therefore\ p_3^2=9$.

*Second.* $$0=v_1+\lambda^2\left(\frac{2.3}{v_0}+\frac{4.5}{v_2}\right)$$
$$=9-p_3^2+\frac{\lambda^2}{2};\quad \therefore\ p_3^2=9+\frac{1}{2}\lambda^2.$$

*Third.* $$0=v_1+\lambda^2\left(\frac{2.3}{v_0}+\frac{4.5}{v_2}\right)+\lambda^4\left[\frac{2.3}{v_0}{}_2S_1+\frac{4.5}{v_2}.{}_3S_1-\frac{2.3}{v_0}.{}_2S_1-\frac{4.5}{v_2}{}_2S_1\right]$$
$$=v_1+\lambda^2\left(\frac{2.3}{v_0}+\frac{4.5}{v_2}\right)-\lambda^4.\frac{4.5.6.7}{v_2^2v_3}$$
$$=9-p_3^2+\lambda^2\left[-\frac{6}{8+\frac{\lambda^2}{2}}+\frac{20}{16-\frac{\lambda^2}{2}}\right]-\frac{4.5.6.7}{16^2.40}\lambda^4;$$
$$\therefore\ p_3^2=9+\frac{1}{2}\lambda^2+\frac{17}{16\times 32}\lambda^4.$$

*Fourth.* $$0=v_1+\lambda^2\left(\frac{2.3}{v_0}+\frac{4.5}{v_2}\right)-\lambda^4.\frac{4.5.6.7}{v_2^2v_3}+\lambda^6.\frac{4.5.6.7}{v_2v_3}\left(\frac{6.7}{v_2v_3}+\frac{8.9}{v_3v_4}\right)$$
$$=9-p_3^2+\lambda^2\left[-\frac{6}{8+\frac{\lambda^2}{2}+\frac{17}{16.32}\lambda^4}+\frac{20}{16-\frac{\lambda^2}{2}-\frac{17}{16.32}\lambda^4}\right]$$
$$-\lambda^4.\frac{4.5.6.7}{\left(16-\frac{\lambda^2}{2}\right)^2\left(40-\frac{\lambda^2}{2}\right)}+\lambda^6.\frac{4.5.6.7}{16^2.40}\left(\frac{6.7}{16.40}+\frac{8.9}{40.55}\right);$$
$$\therefore\ p_3^2=9+\frac{1}{2}\lambda^4+\frac{17}{32\times 16}\lambda^4+\frac{20287}{22\times 16^3\times 100}\lambda^6+\ldots.$$

We have seen if we proceed to solve one fundamental equation in a series of ascending powers of $z\,(\equiv\lambda x)$ we get two separate series, one even and the other odd. But from what we have just done it is clear that these series correspond to *different values of* $p^2$; so that for any particular value of $p^2$ we have really only one solution.

We shall denote this solution by $f(z)$, using suffixes $f_0(z)$, $f_1(z)$, etc. to indicate the values of $p^2$ ($p_0^2 p_1^2 \ldots$) to be taken in the series. One equation is

$$(z^2-\lambda^2)\,y''+zy'+(z^2-p^2)\,y=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

of which we have one solution $y=f(z)$. To get another solution assume $y=vf+w$. Substitute in (1) and we get

$$v\,[(z^2-\lambda^2)f''+zf'+(z^2-p^2)f]+f\,[(z^2-\lambda^2)\,v''+zv']$$
$$+2\,(z^2-\lambda^2)\,v'f'+(z^2-\lambda^2)\,w''+zw'+(z^2-p^2)\,w=0,$$

Since $f$ satisfies equation (1) the coefficient of $v$ in the first line of the last equation vanishes.

Since $v$ is quite at our disposal we may choose it so that

$$(z^2-\lambda^2)\frac{d^2v}{dz^2}+z\frac{dv}{d^2}=0,$$

which is satisfied by $\qquad v=\log(z+\sqrt{(z^2-\lambda^2)}.$

Thus our problem is reduced to finding a *particular* solution (the simpler the better) of the equation

$$(z^2-\lambda^2)\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-p^2)\,w=-2\,(z^2-\lambda^2)\frac{dv}{dz}\frac{df}{dz}=-2\sqrt{z^2-\lambda^2}\frac{df}{dz}\ldots\ldots\ldots\ldots(2).$$

Let $w=\sqrt{z^2-\lambda^2}\,.\,u$. Substitute in (2) and divide by $\sqrt{z^2-\lambda^2}$, and we get

$$(z^2-\lambda^2)\frac{d^2u}{dz^2}+3z\frac{du}{dz}+(z^2-p^2+1)\,u=-2\frac{df}{dz}.$$

We shall obtain $u$ in the form of an ascending series of powers of $z$. The form of the solution will be different for the odd and even series $f_0 f_2 \ldots$ and $f_1 f_3 \ldots$.

Take first the *even* series for $f$.

$$f(z)=a_0+a_1z^2+\ldots+a_nz^{2n}+\ldots,$$

$$(z^2-\lambda^2)\frac{d^2u}{dz^2}+3z\frac{du}{dz}+(z^2-p^2+1)\,u=-2\frac{df}{dz}=-4a_1z+\ldots-4na_nz^{2n-1}+\ldots.$$

Clearly $u$ must be an *odd* series.

Assume $\qquad u=A_0z+A_1z^3+\ldots+A_nz^{2n+1}+\ldots.$

$$(z^2-\lambda^2)\,[6A_1z+20A_2z^3+\ldots+(2n+1)\,(2n)\,A_nz^{2n-1}+\ldots]$$
$$+3z\,[A_0+3A_1z^2+\ldots+(2n+1)\,A_nz^{2n}+\ldots]+(z^2-p^2+1)\,[A_0z+A_1z^3+\ldots+A_nz^{2n+1}+\ldots]$$
$$\equiv-4a_1z+\ldots-4na_n\,z^{2n-1}.$$

Equating the coefficients of different powers of $z$, we get

$$-\;6A_1\lambda^2+(4-p^2)\,A_0=-4a_1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$-20A_2\lambda^2+(16-p^2)\,A_1+A_0=-8a_2;$$
$$-(2n+2)\,(2n+3)\,A_{n+1}\lambda^2+[(2n+2)^2-p^2]\,A_n+A_{n-1}=-4\,(n+1)\,a_{n+1}\ldots(2),$$

which enable us to determine the coefficients successively.

As we require only a particular solution, we may give any values to $A_0$ or $A_1$ we choose, consistently with (1).

We must now examine the convergence of the series $u$.

In the first place we may prove, by proceeding as on p. 47, that $A_n$ cannot be infinite for any *finite* value of $n$.

The relation connecting successive coefficients in $f(z)$ is

$$-(2n+1)(2n+2)\,a_{n+1}\lambda^2+(4n^2-p^2)\,a_n+a_{n-1}=0 \quad\text{.....(1).}$$

If $v_{n+1}=a_{n+1}/a_n$ we have

$$v_{n+1}=\frac{1}{\lambda^2}-\frac{6n+2+p^2}{(2n+1)(2n+2)\lambda^2}+\frac{1}{\lambda^2(2n+1)(2n+2)\,v_n}.$$

Hence *when $n$ is large* we have

*either* $$v_n=-\frac{1}{(2n+1)(2n+2)} \quad\text{.....(2),}$$

*or* $v_n$ is not indefinitely small, and approximates more and more nearly to $\frac{1}{\lambda^2}$ as $n$ increases.

But $p^2$ is chosen so as to make $a_\infty=0$, hence we are confined to the first case (2).

Since then $v_n$ approximates to $-\frac{1}{(2n+1)(2n+2)}$ and $a_n$ to zero, we see that $-4(n+1)\,a_{n+1}$ is indefinitely small for very large values of $n$.

This being the case we see by comparing (1) above with (2) of p. 64, that when $n$ is very large the coefficients of $u$ are connected by the same relation as the coefficients of the convergent series $f$. Hence as the coefficients of $u$ are finite for finite values of $n$, it follows that the series $u$ is convergent in the same domain as the series $f$.

In exactly the same way we may proceed with the odd series for $f$,

$$f(z)=c_0z+c_1z^3+\ldots+c_nz^{2n+1}+\ldots,$$

$$(z^2-\lambda^2)\frac{d^2u}{dz^2}+3z\frac{du}{dz}+(z^2-p^2+1)\,u=-2\frac{df}{dz}$$
$$=-2c_0-6c_1z^2\ldots-2(2n+1)\,c_nz^{2n}\ldots.$$

In this case we take $u=C_0+C_1z^2+\ldots+C_nz^{2n}+\ldots$

and get $$(z^2-\lambda^2)\,[2C_1+12C_2z^2+\ldots+2n(2n-1)\,C_nz^{2n-2}+\ldots]$$
$$+3z\,[2C_1z+4C_2z^3+\ldots+2nC_nz^{2n-1}+\ldots]+(z^2-p^2+1)[C_0+C_1z^2+\ldots+C_nz^{2n}+\ldots]$$
$$\equiv-2c_0-6c_1z^3\ldots-2(2n+1)\,c_nz^{2n}+\ldots.$$

This gives $$-2C_1\lambda^2+(1-p^2)\,C_0=-2c_0$$
$$-12C_2\lambda^2+(9-p^2)\,C_1+C_0=-6c_1$$
$$-(2n+1)(2n+2)\,C_{n+1}\lambda^2+[(2n+1)^2-p^2]\,C_n+C_{n-1}=-2(2n+1)\,c_n,$$

from which, as before, the coefficients may be determined in succession, and the convergence of the series $u$ established.

We have now obtained two independent solutions of our equation, viz.

$$y=f(z) \qquad\qquad (1)$$

$$y=f(z)\log(z+\sqrt{z^2-\lambda^2})+u\sqrt{z^2-\lambda^2}$$
$$=F(z) \qquad\qquad (2).$$

The complete integral is therefore $y=Af(z)+BF(z)$.

The two series are convergent for all *finite* values of the argument. We shall now proceed to obtain two other solutions which represent $y$ 'asymptotically' (to use Poincaré's term)—i.e. we shall obtain two series which approximate more and more closely to solutions of our equation as the argument increases. These series will be very useful for numerical calculation when the argument is not small, and they will also help us to determine what linear function of $f$ and $F$ we must take if we are seeking a solution of our equation which is to vanish when $z=\infty$—a problem that confronts us in many physical applications of our analysis.

Our equation is $$(z^2-\lambda^2)y''+zy'+(z^2-p^2)y=0.$$

Let $y=ue^t$ where $t=iz$ and the equation above becomes

$$(t^2+\lambda^2)u''+(2t^2+t+2\lambda^2)u'+(t-p^2+\lambda^2)u=0.$$

Assuming a serial solution in descending powers of $t$ of the form

$$u=t^m[a_0+a_1/t+a_2/t^2+\dots],$$

we find that we must have $2n+1=0$, therefore $m=-\frac{1}{2}$, and equating coefficients of the various powers of $t$ to zero we get

$$2\frac{a_1}{a_0}=\frac{1}{4}+\lambda^2-p^2,$$

$$4\frac{a_2}{a_0}=\left(\frac{9}{4}+\lambda^2-p^2\right)\frac{a_1}{a_0}-\lambda^2,$$

$$2(n+1)a_{n+1}=\left[\frac{(2n+1)^2}{4}+\lambda^2-p^2\right]a_n-(2n-1)\lambda^2a_{n-1}+(2n-1)(2n+3)\lambda^2a_{n-2},$$

which enable us to determine the coefficients in succession in terms of $a_0$ which is of course arbitrary.

Now let us examine the convergence of the series just obtained.

Let $u_{n+1}=a_{n+1}/a_n$. Then we have

$$2u_{n+1}=\left[n+\frac{\lambda^2-p+1/4}{n+1}\right]-\lambda^2\cdot\frac{2n-1}{n+1}\left[\frac{1}{u_n}-\frac{2n+3}{u_n\cdot u_{n-1}}\right].$$

Hence *when $n$ is large, either* $u_n$ is large, in which case $u_{n+1}=\frac{n}{2}$ nearly, *or* $u_n$ is small, in which case we have approximately when $n$ is large

$$0=1-\frac{2\lambda^2}{n+1}\left[\frac{1}{u_n}-\frac{2n+3}{u_n\cdot u_{n-1}}\right].$$

This relation cannot be satisfied by a *small* value of $u_n$, so that we conclude that when $n$ is large $u_n$ is large, in other words the series is *ultimately divergent*. But, for large values of the argument, the series *begins* by converging rapidly, and so if we stop before the terms begin to diverge the series obtained will be quite well adapted for purposes of calculation. Suppose, for example, that in any numerical problem we agree to neglect terms of the order $\frac{1}{10{,}000}$ and are dealing with values of the argument which make the $n$th term of the above series of the order $\frac{1}{10{,}000}$.

If we stop at the $n$th term of our series and substitute in the differential equation, the equation will not be *quite* satisfied; but instead of having zero on one side of our equation we shall have a few terms of the order $\frac{1}{10{,}000}$. In other words, the series we have taken (stopping at the $n$th term) satisfies our equation *approximately*, the error being of the order we agreed to neglect.

The solution thus obtained is

$$y = \frac{e^{iz}}{\sqrt{z}}\left[a_0 + \frac{a_1}{iz} + \frac{a_2}{(iz)^2} + \dots\right]$$

$$= \frac{e^{iz}}{\sqrt{z}}\left[a_0 - \frac{ia_1}{z} - \frac{a_2}{z^2} - \dots\right] = \phi^+(z) \text{ say.}$$

Changing the sign of $i$ we get another solution, viz.:

$$y = \frac{e^{-iz}}{\sqrt{z}}\left[a_0 + \frac{ia}{z} - \frac{a_2}{z^2} - \dots\right] = \phi^-(z) \text{ say.}$$

The complete integral is $y = C\phi^+(z) + D\phi^-(z)$ where $C$ and $D$ are arbitrary constants.

By taking certain linear functions of $\phi^+$ and $\phi^-$ we build up two solutions that will be of service afterwards. We shall denote these by $\chi^+(z)$ and $\chi^-(z)$, where

$$\chi^+(z) = \frac{\phi^+(z) + \phi^-(z)}{2} = \frac{\cos z}{\sqrt{z}}\left(a_0 - \frac{a_2}{z^2} + \dots\right) + \frac{\sin z}{\sqrt{z}}\left(\frac{a_1}{z} - \frac{a_3}{z^3} + \dots\right)$$

$$= \frac{1}{\sqrt{z}}[R\cos z + S\sin z],$$

where
$$R = a_0 - \frac{a_2}{z^2} + \dots; \quad S = \frac{a_1}{z} - \frac{a_3}{z^3} + \dots,$$

$$\chi^-(z) = i\,.\,\frac{\phi^-(z) - \phi^+(z)}{2} = \frac{1}{\sqrt{z}}[R\sin z - S\cos z].$$

These give $\phi^+ = \chi^+ + i\chi^-$; $\phi^- = \chi^- - i\chi^-$,

$$\phi^+ + i\phi^- = (1+i)(\chi^+ + \chi^-) = \sqrt{2}e^{i\pi/4}(\chi^+ + \chi^-),$$

$$\phi^+ - i\phi^- = (1-i)(\chi^+ + \chi^-) = \sqrt{2}e^{-i\pi/4}(\chi^+ - \chi^-).$$

We proceed to consider the linear relations connecting the functions $f$, $F$, $\phi^+$ and $\phi^-$.

Let $f=\alpha\phi^++\beta\phi^-$ and suppose we are dealing with the *even* series $f$.

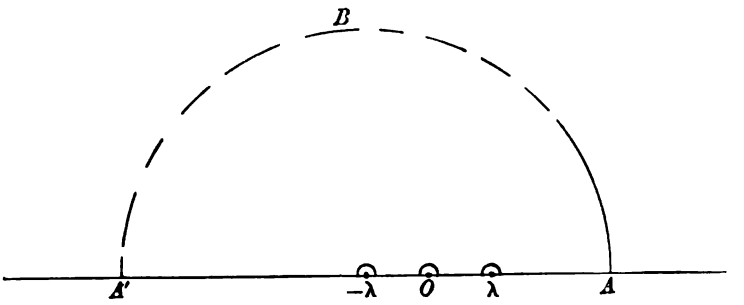
$f(z)$ is a *uniform* and continuous function of $z$, so that if we make a circuit round the origin we shall return to our starting point with the same value of $f$. This however is not the case with $\phi^+$ and $\phi^-$ which are *multiform* functions. It follows then that to insure the continuity of $f$, the constants $\alpha$ and $\beta$ in the above relation must be *discontinuous*.

If we put $z=r(\cos\theta+i\sin\theta)$, we shall find that $\alpha$ must change discontinuously as $\theta$ goes through an odd multiple of $\pi$ and $\beta$ as $\theta$ goes through an even multiple.

We have $e^{iz}=e^{-r\sin\theta}e^{ir\cos\theta}$, so that the modulus of $e^{iz}$ is $e^{-r\sin\theta}$.

But $f=\alpha\phi^++\beta\phi^-=\alpha e^{-r\sin\theta}(\ldots)+\beta e^{r\sin\theta}(\ldots)$ and, when $\theta$ lies between 0 and $\pi$, $\sin\theta$ is positive, so that $\phi^-$ is (for large values of $r$ to which we are confined when dealing with the functions $\phi$) exceedingly large compared with $\phi^+$, and so is far more important than $\phi^+$. But when $\theta$ passes through $\pi$ these relations are reversed. The term $\phi^+$ is now the all important one. Hence as $\theta$ passes through $\pi$ the constant $\alpha$ must change discontinuously to insure the continuity of $f$. There is no possibility of another discontinuity till we get to $\theta=2\pi$, $\phi^-$ must now become the important term so that $\beta$ must change abruptly, and so on. We conclude then that $\alpha$ changes discontinuously as $\theta$ passes through an odd multiple of $\pi$, and $\beta$ when $\theta$ goes through an even multiple.

We can obtain a relation between $\alpha$ and $\beta$ by working on the same lines as on p. 52.




If we start with any of our functions at $A$ (for which $\theta=0$) and go along the circle of very great radius $ABA'$ to $A'(\theta=\pi)$, we must get to the same result as if we go along $AOA'$ [avoiding the points $\lambda$, $O-\lambda$ by describing small semi-circles round them], for the space enclosed by these two paths contains no critical points of our functions.

Suppose then we take $f$ along $AOA'$ and its equivalent $\alpha\phi^++\beta\phi^-$ along the great circle.

Then $f$ is valid all along its path and so is $\alpha\phi^++\beta\phi^-$, the functions $\phi$ being defined only for very large values of the argument.

Starting from $A$ and going to $A'$ we increase $\theta$ by $\pi$. $f$ is even and so does not change sign. Hence we reach $A'$ with $f$; but $f$ at $A'$ is not equal to $\alpha\phi^+ + \beta\phi^-$ but to $\alpha'\phi^+ + \beta\phi^-$ (since $\alpha$ is discontinuous at $\theta = \pi$). For brevity we shall denote $f(z) = f(r.\overline{\cos\theta + i\sin\theta})$ by $f(\theta)$.

Thus we get
$$\begin{aligned}\alpha\phi^+(0) + \beta\phi^-(0) = f(0) &= f(\pi)\\ &= \alpha'\phi^+(\pi) + \beta\phi^-(\pi)\\ &= -i\alpha'\phi^-(0) - i\beta\phi^+(0).\end{aligned}$$

Hence
$$\alpha = -i\beta \text{ and } \beta = -i\alpha'.$$

We have found then that when $\pi > \theta > 0$, $f = \alpha(\phi^+ + i\phi^-)$, and in exactly the same way we see that for the *odd* function we have $f = \alpha_1(\phi^+ - i\phi^-)$.

Now consider the other function
$$F = u\sqrt{z^2 - \lambda^2} + f\log(z + \sqrt{z^2 - \lambda^2}).$$
To make this definite we may take $\sqrt{z^2 - \lambda^2} = +\sqrt{\mu}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ where $\mu$ is the modulus and $\theta$ the amplitude of $z^2 - \lambda^2$; and for $\log(z + \sqrt{z^2 - \lambda^2})$ take its principal value.

Suppose first $f$ is even, then $u$ is odd (see p. 64).

Let $F = A\phi^+ + B\phi^-$ and carry out the process of last page. On going round $z = \lambda$, $\sqrt{z^2 - \lambda^2}$ becomes $i\sqrt{z^2 - \lambda^2}$ and so $F$ becomes $iu\sqrt{z^2 - \lambda^2} + f\log(z + i\sqrt{z^2 - \lambda^2})$. Passing $z = 0$ $z$ becomes negative, and $u$ being odd becomes $-u$; while $f$ is unaffected. Thus $F$ has become $-iu\sqrt{z^2 - \lambda^2} + f\log(-z + i\sqrt{z^2 - \lambda^2})$, where of course $z$ here means $|z|$.

Then going round $-\lambda$, $F$ becomes $u\sqrt{z^2 - \lambda^2} + f\log(-z - \sqrt{z^2 + \lambda^2}) = F + \pi i f$, where $F$ means the value with which we started.

Just as before $A$ must be a discontinuous constant, changing as $\theta$ goes through an odd multiple of $\pi$ and $B$ as $\theta$ goes through an even multiple.

If $A$ becomes $A'$ when $\theta = \pi$ we then have:—
$$\begin{aligned}A'\phi^+(\pi) + B\phi^-(\pi) &= F(0) + i\pi f(0)\\ &= A\phi^+(0) + B\phi^-(0) + i\pi\alpha(\phi^+ + i\phi^-);\end{aligned}$$
$$\therefore\ -iA'\phi^-(0) - iB\phi^+(0) = A\phi^+(0) + B\phi^-(0) + i\pi\alpha(\phi^+ + i\phi^-);$$
$$\therefore\ -iA' = B - \pi\alpha;\ -iB = A + \pi i\alpha,$$
and
$$F = A(\phi^+ + i\phi^-) - \pi\alpha\phi^-,\ [\pi > \theta > 0].$$

Treating the *odd* series $f$ in the same way we get
$$F = A'(\phi^+ - i\phi^-) + \pi\alpha_1\phi^-,\ [\pi > \theta > 0].$$

We have now only *two* undetermined constants, and these can be easily found by calculating the different series for *one* value of the argument. But for a large and important class of problems even this labour is not required. In these cases all that is wanted is the roots of $f(z)=0$ or $f'(z)=0$, or something of this form. The larger roots in these cases necessitate the use of the functions $\phi$, but it is at once evident that the value of the constant $\alpha$ is not required. We shall now investigate the larger roots of the equation $f(z)=0$. Taking the even series $[f_0, f_2 \ldots]$ we see from last page that our equation is $\phi^+(z)+i\phi^-(z)=0$, or $\chi^+(z)+\chi^-(z)=0$,

or
$$R\cos\left(z-\frac{\pi}{4}\right)+S\sin\left(z-\frac{\pi}{4}\right)=0.$$

If
$$\tan\psi=S/R=\frac{\dfrac{a_1}{z}-\dfrac{a_3}{z^3}+\ldots}{a_0-\dfrac{a_2}{z^2}+\ldots}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

then our equation becomes $\cos\left(z-\frac{\pi}{4}-\psi\right)=0$, and $z-\frac{\pi}{4}-\psi=(2m+1)\frac{\pi}{2}$ where $m$ is an integer, so that

$$z=m\pi+\frac{3\pi}{4}+\psi\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

When $z$ is *very large* we see from (1) that $\psi=0$, and thus the very large roots are given by $z=m\pi+\frac{3\pi}{4}$. $\psi$ diminishes as $z$ increases, so that the difference between two consecutive roots is more and more nearly equal to $\pi$ as the sign of the root increases.

In the general case having got $\tan\psi$ from (1) in terms of $z$ we use the expansion $\psi=\tan\psi-\frac{1}{3}\tan^3\psi+\frac{1}{5}\tan^5 4\ldots$, substitute in (2) and proceed by successive approximations.

The treatment of the odd series $[f_1, f_3 \ldots]$ is of course precisely similar. Our equation is now $\phi^+(z)-i\phi^-(z)=0$, i.e. $\chi^+(z)-\chi^-(z)=0$, or

$$R\sin\left(z-\frac{\pi}{4}\right)-S\cos\left(z-\frac{\pi}{4}\right)=0,$$

$$\sin\left(z-\frac{\pi}{4}-\psi\right)=0,\quad\therefore\ z-\frac{\pi}{4}-\psi=m\pi,$$

where $m$ is an integer, so that

$$z=m\pi+\frac{\pi}{4}+\psi.$$

In exactly the same way we may deal with the larger roots of $f'(z)=0$.

In most physical applications, the root of most importance is the *lowest* root; and it will sometimes happen that that root is too small to make the process just con-

sidered effective. In such a case there is no difficulty in finding the root directly from the *ascending* series, which converges rapidly (the argument being small).

We shall illustrate this by finding the lowest root of $f_0(z)=0$.

We have $p_0^2=\frac{\lambda^2}{2}-\frac{\lambda^4}{32}-\dots$ (up to $\lambda^8$)... and using this value of $p^2$ we get

$$\frac{a_1}{a_0}=-\frac{1}{4}+\frac{\lambda^2}{64}+\dots; \qquad \frac{a_2}{a_0}=\frac{1}{64}-\frac{\lambda^2}{12.64}+\dots;$$

$$\frac{a_3}{a_0}=-\frac{1}{(6.8)^2}+\frac{7\lambda^2}{90\times 64\times 30}+\dots. \qquad \frac{a_4}{a_0}=\frac{1}{(2.4.6.8)^2}+\dots.$$

Hence
$$f_0(z)=1-z^2\left[\frac{1}{2^2}-\frac{\lambda^2}{64}+\dots\right]+z^4\left[\frac{1}{(2.4)^2}-\frac{\lambda^2}{12.64}+\dots\right]$$
$$-z^6\left[\frac{1}{(2.4.6)^2}-\frac{7\lambda^2}{96.64.30}+\dots\right]+z^8\left[\frac{1}{(2.4.6.8)^2}+\dots\right]+\dots$$

Now $z=\lambda x=\kappa a$; $\lambda=\kappa a e$, so that $f_0(z)=0$ is equivalent to

$$0=1-z^2.\frac{1}{2^2}+z^4\left[\frac{1}{(2.4)^2}+\frac{e^2}{64}\right]-z^6\left[\frac{1}{(2.4.6)^2}+\frac{e^2}{12.64}\right]$$
$$+z^8\left[\frac{1}{(2.4.6.8)^2}+\frac{7e^2}{96.64.30}\right]+\dots \qquad \dots\dots\dots\dots\dots\dots\dots\dots(1).$$

The series on the right of (1) is rapidly convergent, and the terms are alternately positive and negative. If we stop at the $n$th term we get an algebraic equation which (for different values of $e$) can be solved by Horner's process with very little difficulty if $n$ is not very great. The root thus obtained will not of course be exactly a root of $f_0(z)=0$, but it will be a close approximation if $n$ is not too small. Also it is clear that the roots of the equation corresponding to $n$ and $n+1$ will be the one greater and the other less than the root of $f_0(z)=0$. Thus by solving the equations corresponding to $n$ and $n+1$ and taking $z$ to lie between the two roots we shall get a close approximation to the real root.

Retaining the first five terms of our series and putting $y=\frac{1}{z^2}$ we have to solve:—

$$y^4-\cdot 25y^3+y^2[\cdot 015625+\cdot 015625e^2]$$
$$-y[\cdot 000434+\cdot 001302e^2]$$
$$+[\cdot 000006781+\cdot 000038e^2]=0.$$

For $e=0$, the equation becomes

$$y^4-\cdot 25y^3+\cdot 015625y^2-\cdot 000434y+\cdot 000006781=0.$$

Attacking this by Horner's process we get

| | | | | |
|---|---|---|---|---|
| 1 | −2·5 | 1·5625 | −·434 | ·0678 ( ·1728 |
| | 1 | −1·5 | ·0625 | − 3715 |
| | −1·5 | ·0625 | −·3715 | − 3037 |
| | 1 | − ·5 | −·4375 | 2178·75 |
| | − ·5 | − ·4375 | − 809 | − 8582 500 |
| | 1 | ·5 | 1121·75 | 5957 556 |
| | ·5 | 6·25 | 311·25 | − 2624 944 |
| | 1 | 154 | 2542·75 | |
| | 1·5 | 160·25 | 2854 000 | |
| | 7 | 203 | 124 778 | |
| | 22 | 363·25 | 2978 778 | |
| | 7 | 252 | 126514 | |
| | 29 | 615 25 | 3105292 | |
| | 7 | 864 | | |
| | 36 | 62389 | | |
| | 7 | 868 | | |
| | 430 | 63257 | | |
| | 2 | | | |
| | 432 | | | |
| | 2 | | | |
| | 434 | | | |

The equation $y^3-\cdot 25y^2+\cdot 015625y-\cdot 000434=0$ gives in the same way $y=\cdot 1748$. Thus the real root must lie between ·1728 and ·1748. If we go to a higher order we find $y=\cdot 1730$ as a very close approximation.

Since $y=\frac{1}{z^2}$ this makes $z=2\cdot 404$.

Treating the equation corresponding to other values of $e$ in the same way we get these results:—

For $e=0,\ z=2\cdot 404$; $e=0\cdot 4,\ z=2\cdot 512$.
$e=0\cdot 1,\ z=2\cdot 411$; $e=0\cdot 5,\ z=2\cdot 585$.
$e=0\cdot 2,\ z=2\cdot 429$; $e=0\cdot 6,\ z=2\cdot 774$.
$e=0\cdot 3,\ z=2\cdot 457$; $e=0\cdot 7,\ z=3\cdot 092$.

For larger values of $e$, we should take in more terms of our series to get a close approximation to the root, which is getting too large to make our approximation very accurate. For these larger roots it is better to use the *descending* series, and the method of finding the roots from them, explained on p. 70.

We shall proceed to consider briefly several physical problems that can be solved by the aid of the functions we have been considering.

*Vibrations of elliptic membranes:*

If $T$ is the tension, $\rho$ the density and $w$ the small displacement normal to the plane of the membrane, the equation of motion is

$$\frac{d^2w}{dt^2}=\frac{T}{\rho}V_1^2w=c^2V_1^2w \text{ where } T=\rho c^2.$$

Taking $w$ to vary as $e^{i\kappa ct}$, we get $(V_1^2+\kappa^2)w=0$. Since $w$ must be finite all over the membrane we are confined to the $f$ functions—supposing the membrane to be a complete ellipse.

Thus we take $$w=\Sigma A_n f_n(\lambda x) f_n(\lambda x')\, e^{i\kappa ct}.$$

$\kappa$ (on which the frequency depends) is determined by the boundary condition; which is $w=0$ when $x=x_0$.

This gives $f_n(\lambda x_0)=0$, or $f_n(\kappa a)=0$, where $a$ is the semi-major axis of the ellipse.

The nodal system is composed of a series of confocal ellipses given by $f_n(\lambda x)=0$, and a series of confocal hyperbolas $f_n(\lambda x')=0$.

To determine the frequencies corresponding to the various fundamental modes of vibration we have to find the lowest roots of $f_0(z)=0$, $f_1(z)=0$, and so on, and we have already seen how to do this.

Taking the values found for these roots in the case of $f_0(z)=0$, we find that the ratios of the frequency of the fundamental note for $e=0$, $e=0{\cdot}1$, $e=0{\cdot}2$, $e=0{\cdot}3$, ... $e=0{\cdot}7$ to that for $e=0$ are 1; 1·003; 1·013; 1·022; 1·045; 1·075; 1·154; 1·286. Thus there is very little difference between the notes emitted in the different cases. The interval between the notes for $e=0$ and $e=0{\cdot}1$ is less than a *comma*; between $e=0$ and $e=0{\cdot}2$ just about a comma, between $e=0{\cdot}6$ and $e=0$ about a *minor third*; and between $e=0$ and $e=0{\cdot}7$ an interval between a minor third and a fourth. Of course the frequency rises as the eccentricity increases.

*The vibrations of an elliptic plate* can also be determined. If $E$ be Young's modulus for the material, $\rho$ the density, $2t$ the thickness, and $\mu$ Poisson's ratio, then if $w$ is the displacement normal to the plane of the plate the dynamical equation is $\ddot{w}+c^4V^4w=0$, where $c^4=\dfrac{Et^2}{3\rho(1-\mu^2)}$. If as usual we take $w$ to vary as $e^{ipt}$ and put $\kappa^4=p^2/c^4$ our equation becomes $(V^4-\kappa^4)w=0$, or $(V^2+\kappa^2)(V^2-\kappa^2)w=0$. This can be solved in terms of our functions, and as $w$ must be finite all over the plate we are confined to the $f$ functions.

We may take then $w=Ae^{i\kappa^2c^2t}[f(z)+\mu'f(iz)][f(z')+\mu'f(iz^2)]$, where $z=\lambda x$; $z'=\lambda x'$.

The nodal system consists of the series of confocal ellipses given by $f(z)+\mu'f(iz)=0$, and the series of confocal hyperbolas given by $f(z')+\mu'f(iz')=0$.

If the plate is *clamped at the edge* ($x=x_0$), we must have $w$ and $\dfrac{dw}{dz}=0$ at the edge,

$$\therefore\ f(z_0)+\mu'f(iz_0)=0,$$
$$f'(z_0)+i\mu'f(iz_0)=0.$$

Eliminating $\mu'$ we get the frequency equation $[z_0=\kappa a]$

$$\frac{f'(z_0)}{f(z_0)}=\frac{if'(iz_0)}{f(iz_0)}.$$

*Vibration of air in elliptic cylinders.*

If $\psi$ is the velocity potential we have as usual $\frac{d^2\psi}{dt^2}=c^2V_1^2\psi$, and taking $\psi$ to vary as $e^{i\kappa ct}$ we get $(V_1^2+\kappa^2)\psi=0$.

As $\psi$ must be finite at all points in the cylinder we take

$$\psi=\Sigma A_n f_n(\lambda x) f_n(\lambda x')\, e^{i\kappa ct}.$$

Since there is no normal displacement at the boundary we have $\frac{\partial\psi}{\partial x}=0$ when $x=x_0$, so that corresponding to $\psi=A_n e^{i\kappa ct} f_n(\lambda x) f_n(\lambda x')$ we have the frequency equation $f_n'(\lambda x_0)=0$, or $f_n'(\kappa a)=0$. We have already explained (p. 70) how the roots of this equation are to be obtained from the *descending* series.

We shall write down the results for the lowest note of the modes whose velocity potentials are

$$\psi=Af_0(\lambda x) f_0(\lambda x')\, e^{i\kappa ct},$$

and

$$\psi=A_1 f_1(\lambda x) f_1(\lambda x')\, e^{i\kappa ct}.$$

Any other mode may of course be discussed by the same method.

Mode $\psi=Af_0(\lambda x) f_0(\lambda x')\, e^{i\kappa ct}$.

($c=331{,}00$ centimetres per second.)

| $e$ | $\kappa a$ | $a\times$ frequency | wave length/$a$ | Ratio of frequency to that for $e=0$ |
|---|---|---|---|---|
| 0 | 3·832 | 20156 | 1·6400 | 1 |
| 0·1 | 3·8432 | 20221 | 1·6346 | 1·0033 |
| 0·2 | 3·8720 | 20366 | 1·6230 | 1·0103 |
| 0·3 | 3·9259 | 20650 | 1·6007 | 1·0245 |
| 0·4 | 4·0578 | 21346 | 1·5495 | 1·0591 |
| 0·5 | 4·1874 | 22025 | 1·5008 | 1·0927 |
| 0·6 | 4·3799 | 23036 | 1·4348 | 1·1430 |
| 0·7 | 4·7328 | 24895 | 1·3277 | 1·2351 |
| 0·8 | 5·2923 | 27835 | 1·1875 | 1·4132 |

We see from the above that there is an interval of rather less than a comma between the notes emitted in the case $e=0$ and in $e=0{\cdot}2$. If $e=0$ gives out the note c then $e=0{\cdot}8$ will sound F♯ very nearly.

In order that $e=0$ may omit the middle c of a piano (264 vibrations a second) its radius must be

$$a = 76{\cdot}35 \text{ centimetres} = 30{\cdot}061 \text{ inches}.$$

$$\text{Mode } \psi = Af_1(\lambda x) f_1(\lambda x') e^{i\kappa ct}.$$

[The lowest roots of $f_1'(\lambda x)$ is not very large and is best obtained by the method of p. 71.]

| $e$ | $\kappa a$ | $a \times$ frequency | wave length/$a$ | Ratio of frequency to that for $e=0$ |
|---|---|---|---|---|
| 0 | 1·8410 | 9683 | 3·4135 | 1 |
| 0·1 | 1·8417 | 9687 | 3·4120 | 1·0005 |
| 0·2 | 1·8421 | 9690 | 3·4112 | 1·0007 |
| 0·3 | 1·8449 | 9705 | 3·4057 | 1·0024 |
| 0·4 | 1·8494 | 9728 | 3·3979 | 1·0047 |
| 0·5 | 1·8527 | 9744 | 3·3900 | 1·0063 |
| 0·6 | 1·8576 | 9772 | 3·3823 | 1·0093 |
| 0·7 | 1·8657 | 9813 | 3·3683 | 1·0135 |
| 0·8 | 1·8758 | 9867 | 3·3496 | 1·0191 |
| 0·9 | 1·8875 | 9929 | 3·3290 | 1·0255 |

This is the gravest of the normal modes. The influence of eccentricity on the frequency is exceedingly minute.

If the motion is not the same in every transverse plane we can still solve the problem as indicated on p. 42. Thus suppose, for example, the cylinder is bounded by rigid transverse walls at $z=0$ and $z=l$, then we take

$$\psi = \Sigma A_n \cos\frac{m\pi z}{l},$$

and get

$$(V_1^2 + \kappa'^2) A_n = 0,$$

where

$$\kappa'^2 = \kappa^2 - \frac{m^2\pi^2}{l^2},$$

so that we have finally

$$\psi = \Sigma B_n . f_n(\lambda' x) f_n(\lambda' x') \cos\frac{m\pi z}{l} e^{i\kappa ct},$$

where $\lambda' = h\kappa'$ and $\kappa'$ is given above.

The *lowest* note corresponds to $m=1$, and

$$\kappa^2=\kappa'^2+\frac{\pi^2}{l^2},$$

or

$$(\kappa a)^2=(\kappa' a)^2+\frac{\pi^2 a^2}{l^2} \quad \text{.....(1).}$$

The lowest values of $\kappa' a$ for the different cases have just been written down for the case $n=0$, and the corresponding frequencies are thus obtainable with the aid of (1).

Another problem that is readily solved by the help of our functions is the determinations of the *tidal waves in a cylindrical vessel* of elliptical boundary.

If $\xi$ be the elevation of the free surface above the undisturbed level then it is well known that 'the equation of continuity' and the dynamical equations lead at once to $\frac{d^2\xi}{dt^2}=c^2V_1^2\xi$ where $c^2=gh$, ($h$ being the depth).

Thus we have $\zeta=\Sigma A_n f_n(\lambda x)f_n(\lambda x')e^{i\kappa ct}$, the boundary equation being $\frac{d\zeta}{dn}=0$ or $f_n'(\lambda x_0)=0$. This is the frequency equation, giving the admissible values of $\kappa$, and the corresponding 'speeds' of the oscillations are then $\kappa c=\kappa\sqrt{gh}$.

Another hydrodynamical problem that naturally presents itself here is the consideration of certain possible forms of *steady vortex motion in an elliptic cylinder*. In steady motion in two dimensions the angular velocity ($\omega$) is constant along each stream line. But if $\psi$ be the stream function we have $V_1^2\psi=2\omega$, so that every possible form of steady motion is included in $V_1^2\psi=\chi(\psi)$ where $\chi(\psi)$ is an arbitrary function of $\psi$. If we put $\chi(\psi)=-\kappa^2\psi$ where $\kappa$ is a constant we get our equation $(V_1^2+\kappa^2)\psi=0$, and as $\psi$ must be finite at all points within the cylinder we have

$$\psi=\Sigma A_n f_n(\lambda x)f_n(\lambda x').$$

As $\psi$ must be constant along the boundary, we get for the case corresponding to $\psi=Af_n(\lambda x)f_n(\lambda x')$, $f_n(\lambda x_0)=0$ or $f_n(\kappa a)=0$, the roots of which determine the admissible values of $\kappa$. We have already shown how to obtain these roots and have written down the lowest root for $n=0$, corresponding to various values of the eccentricity, (p. 72).

The *periods of vibration of electricity* in a cylindrical cavity of elliptic cross-section inside a conductor are readily obtained.

Since everything is independent of $z$ (measured along the axis) we have, with usual notation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$; thus there is a stream function $\psi$, from which $u$ and $v$ are got by differentiation $u=\frac{\partial\psi}{\partial y}$; $v=-\frac{\partial\psi}{\partial x}$. By applying the circuital laws of Ampère and Faraday in the usual way we get $\frac{\partial^2\psi}{\partial t^2}=V^2(V_1^2\psi)$ in the dielectric, where $V$ is the velocity of propagation of electro-dynamic action through the dielectric.

Putting $\psi \,.\, \propto e^{i\kappa Vt}$ we get $(V_1^2+\kappa^2)\psi=0$, and since $\psi$ must be finite everywhere in the cavity, we thus get

$$\psi=\Sigma A_n f_n(\lambda x) f_n(\lambda x') e^{i\kappa Vt}.$$

Now the wave length will clearly be comparable in magnitude with the diameter of the cylinder, so that for ordinary sized cylinders the frequency is enormously high. But in such cases the currents are confined to a mere 'skin' on the surface of the conductor. Inside this skin (i.e. in the conductor) there is no E.M.F.; and the tangential E.M.F. is continuous on crossing the skin. Thus the tangential E.M.F. must vanish at the surface of the dielectric. Since the tangential E.M.F. vanishes, the tangential current also vanishes, therefore $\frac{\partial\psi}{\partial n}=0$, at the boundary, or in our notation $\frac{\partial\psi}{\partial x}=0$, when $x=x_0$.

Thus if $\psi=A f_n(\lambda x) f_n(\lambda x') e^{i\kappa Vt}$, the admissible values of $\kappa\,(\lambda=h\kappa)$ are given by the roots of $f_n'(\lambda x_0)=0$. The roots of this are given (for $n=0$ and $n=1$) on pp. 74, 75; the wave lengths there determined apply also to the electrical problem.

If we are dealing with the problem of electric waves in the dielectric surrounding an elliptic cylinder then we must replace $f_n(\lambda x)$ above by $\phi_n^-(\lambda x)$ and take $\psi=\Sigma A_n \phi_n^-(\lambda x) f_n(\lambda x')\, e^{i\kappa Vt}$ in the dielectric. In this case the admissible values of $\kappa$ are given by the roots the equation $\phi_n'^-(\kappa a)=0$.

If we wish to estimate the *decay of magnetic force in a metal cylinder*, the lines of magnetic force being parallel to the axis we have (neglecting polarisation currents) $V_1^2 c=\frac{4\pi\mu}{\sigma}\frac{dc}{dt}$ or, if $c\,.\propto e^{-mt}$, $(V_1^2+\kappa^2)c=0$ where $\kappa^2=\frac{4\pi\mu m}{\sigma}$.

The appropriate solution is $c=\Sigma A_n f_n(\lambda x) f_n(\lambda x')\, e^{-mt}$. Now we are neglecting polarisation currents so that $c$ is constant in the dielectric. Taking this constant value as $I$, and noting that the tangential component of the magnetic force is continuous, we have at the boundary $(x=x_0)$

$$\Sigma A_n f_n(\lambda x_0) f_n(\lambda x')\, e^{-mt}=\mu I.$$

In the case of *free currents* $I=0$ and the admissible values of $\kappa$ are given by the roots of $f_n(\lambda x_0)=0$, or $f_n(\kappa a)=0$.

The 'modulus of decay' is $\frac{1}{m}=\frac{4\pi m}{\sigma\kappa^2}=\frac{4\pi\mu a^2}{\sigma(\kappa a)^2}$. The decay is slowest for the smallest values of $(\kappa a)$, the slowest of all corresponding to the least root of $f_0(\kappa a)=0$. We shall write down the modulus for a copper rod semi-major axis 1 *cm*, for which $\sigma=1600$; and for an iron rod of same size for which $\mu=1000$, and $\sigma=10^4$.

| $e$ | $\kappa a$ | Modulus: Copper (seconds) | Modulus: Iron (seconds) | Ratio to case $e=0$ |
|---|---|---|---|---|
| 0 | 2·404 | ·001359 | ·2174 | 1 |
| 0·1 | 2·411 | ·001351 | ·2162 | 0·9942 |
| 0·2 | 2·429 | ·001331 | ·2130 | 0·9793 |
| 0·3 | 2·457 | ·001301 | ·2082 | 0·9573 |
| 0·4 | 2·512 | ·001245 | ·1991 | 0·9158 |
| 0·5 | 2·585 | ·001175 | ·1881 | 0·8649 |
| 0·6 | 2·774 | ·001021 | ·1633 | 0·7510 |
| 0·7 | 3·092 | ·000803 | ·1284 | 0·5907 |

In dealing with the case where the *currents are longitudinal* and the magnetic force transversal, we may express everything in terms of $R$ the E.M.F. along the axis. Adopting the usual notation we have:—

$$\frac{da}{dt}=-\frac{\partial R}{\partial y}; \quad \frac{\partial b}{dt}=\frac{\partial R}{\partial x}; \quad c=0; \quad u=v=0;$$

$$4\pi\mu ip\omega=4\pi\mu\frac{dw}{dt}=V_1^2R.$$

In the cylinder we have $(V_1^2+\kappa_1^2)R_1=0$ where $\kappa_1^2=-\dfrac{4\pi\mu ip}{\sigma}$, and in the dielectric $(V_1^2+\kappa_0^2)R_0=0$ where $\kappa_0^2=\left(\dfrac{p}{V}\right)^2$.

Thus
$$R_1=\Sigma A_n f_n(\lambda_1 x)f_n(\lambda_1 x')e^{ipt},$$
$$R_0=\Sigma B_n\phi_n^-(\lambda_0 x)f_n(\lambda_0 x')e^{ipt}.$$

The E.M.F. parallel to the axis is continuous at the surface of the cylinder, $\therefore\ R_1=R_0$ when $x=x_0$, $\therefore\ A_n f_n(\lambda_1 x_0)=B_n\phi_n^-(\lambda_0 x_0)$. $R$ is a 'stream function' for $a$ and $b$, so that the magnetic induction parallel to the bounding ellipse is proportional to $\dfrac{\partial\psi}{\partial x}$.

But the tangential magnetic force is continuous; so that we get another boundary equation $\dfrac{A_n\lambda_1}{\mu}f_n'(\lambda_1 x_0)=B_n\lambda_0\phi_n'^-(\lambda_0 x_0)$.

Eliminating the ratio of $A_n$ to $B_n$ we get:—

$$\frac{\lambda_1 f_n'(\lambda_1 x_0)}{\mu f_n(\lambda_1 x_0)}=\frac{\lambda_0\phi_n'^-(\lambda_0 x_0)}{\phi_n^-(\lambda_0 x_0)} \quad\text{.....} \quad (1).$$

Since the currents will decay slowly $p/V$ (and therefore $\lambda_0$) is very small.

Now if $\lambda = 0$, the equation for $\phi_n^-(\lambda x)$ is $z^2 \dfrac{dy^2}{dz^2} + z\dfrac{dy}{dz} + (z^2 - n^2)\, y = 0$.

Let $y = ue^t$ where $t = -iz$ and we get $t^2 \dfrac{d^2u}{dt^2} + (2t^2 + t)\dfrac{du}{dt} + (t - n^2)\, u = 0$.

Solving this in an *ascending* series of the form $\dfrac{a_m}{t^m} + \dfrac{a_{m+1}}{t^{m-1}} + \ldots$, we get as the "indicial" equation $m^2 - n^2 = 0$.

Thus we have $\phi_n^-(\lambda x) = \dfrac{ae^{-i\lambda x}}{(\lambda x)^n}$ approximately when $\lambda$ is small.

Making use of this we find that (1), of p. 78, becomes

$$\frac{\lambda_1 f_n'(\lambda_1 x_0)}{\mu f_n(\lambda_1 x_0)} = -\, n\lambda_0,$$

or

$$\lambda_1 f_n'(\lambda_1 x_0) + n\lambda_0 \mu f_n(\lambda_1 x_0) = 0,$$

which determines the admissible values of $p$. If, as in case of iron, $\mu$ is large we have $f_n(\lambda_1 x_0) = 0$, which has just been considered for the case $n = 0$ on p. 78.

## Section II. Spheroids.

We shall proceed now to the more interesting problem of finding solutions of the equation $(V^2 + \kappa^2)\psi = 0$ that will enable us to deal with a variety of questions relating to spheroids.

Generally if $\alpha_1$, $\alpha_2$, $\alpha_3$ form a system of orthogonal coordinates so that the line element $ds$ is given by $ds^2 = \dfrac{d\alpha_1^2}{h_1^2} + \dfrac{d\alpha_2^2}{h_2^2} + \dfrac{d\alpha_3^2}{h_3^2}$, it is well known that

$$V^2\psi = h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha_1}\left(\frac{h_1}{h_2 h_3}\frac{\partial \phi}{\partial \alpha_1}\right) + \ldots + \ldots\right].$$

Also if $\alpha_1$, $\alpha_2$, $\alpha_3$ satisfy $V^2\psi = 0$, we have the simpler relation

$$V^2\psi = h_1^2 \frac{\partial^2 \psi}{\partial \alpha_1^2} + h_2^2 \frac{\partial^2 \psi}{\partial \alpha_2^2} + h_0^2 \frac{\partial^2 \psi}{\partial \alpha_0^2}.$$

We shall be dealing with prolate spheroids, and shall define the position of a point by the semi-axes $a$ and $a'$ of the ellipse and hyperbola confocal with some given one that pass through the point, and by the azimuth $\phi$.

We may take $\alpha_1 = -\frac{1}{2}\displaystyle\int_a^\infty \frac{d\lambda}{\sqrt{(a_0^2 + \lambda)(b_0^2 + \lambda)}}$ (with the usual notation).

$$= -\int_a^\infty \frac{da}{b^2}, \text{ so that } d\alpha_1 = \frac{da}{b^2},$$

$$pdp = ada;\ h_1 = \frac{d\alpha_1}{dn} = \frac{p}{a}\frac{\partial a}{\partial a} = \frac{p}{ab^2} = \frac{1}{bD} = \frac{1}{b\sqrt{a^2 - a'^2}}.$$

In this way we get $V^2\psi=\dfrac{1}{a^2-a'^2}\left[\dfrac{1}{b^2}\dfrac{\partial^2\psi}{\partial\alpha_1^2}+\dfrac{1}{b'^2}\dfrac{\partial^2\psi}{\partial\alpha_2^2}+\left(\dfrac{1}{1-\dfrac{a'^2}{h^2}}-\dfrac{1}{1-\dfrac{a^2}{h^2}}\right)\dfrac{\partial^2\psi}{\partial\phi^2}\right]$;

i.e. $V^2\psi=\dfrac{1}{h^2(x^2-x'^2)}\left[-\dfrac{\partial}{\partial x}(1-x^2)\dfrac{\partial\psi}{\partial x}+\dfrac{\partial}{\partial x'}(1-x'^2)\dfrac{\partial\psi}{\partial x'}+\left(\dfrac{1}{1-x'^2}-\dfrac{1}{1-x^2}\right)\dfrac{\partial^2\psi}{\partial\phi^2}\right]$,

where $x=a/h$ and $x'=a'/h$.

If we take $\psi$ to vary as $\sin(n\phi+\gamma)$ we have $\dfrac{\partial^2\psi}{\partial\phi^2}=-n^2\psi$,

and hence if $(V^2+\kappa^2)\psi=0$ we get

$$h^2\kappa^2(x^2-x'^2)\psi=\frac{\partial}{\partial x}(1-x^2)\frac{\partial\psi}{\partial x}-\frac{\partial}{\partial x'}(1-x'^2)\frac{\partial\psi}{\partial x'}+\left(\frac{1}{1-x'^2}-\frac{1}{1-x^2}\right)n^2\psi.$$

As before assume $\psi=yy'$ where $y$ is a function of $x$ only and $y'$ of $x'$ only. We then get (putting $h\kappa=\lambda$)

$$\frac{1}{y}\frac{d}{dx}(1-x^2)\frac{dy}{dx}-\frac{n^2}{1-x^2}-\lambda^2x^2=\frac{1}{y'}\frac{d}{dx'}(1-x'^2)\frac{dy'}{dx^2}-\frac{n^2}{1-x'^2}-\lambda^2x'^2$$

$=-p$ say (where $p$ is an arbitrary constant).

Thus $y$ and $y'$ both satisfy an equation of the form

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)\,y-\frac{n^2}{1-x^2}\,y=0.$$

In case of symmetry about the axis we have $n=0$ and

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)\,y=0.$$

Consider $$(1-x^2)\,y''-2xy'+(p-\lambda^2x^2)\,y-\frac{n^2}{1-x^2}\,y=0 \quad\text{..... (1).}$$

Let $y=(1-x^2)^{n/2}.\,z$. Substitute in (1) and divide by $(1-x)^{n/2}$.

This leads to the equation

$$(1-x^2)\,z''-2(n+1)\,xz'+[p-n(n+1)-\lambda^2x^2]\,z=0 \quad\text{..... (2).}$$

Everything now depends on the solution of this equation which we may therefore regard as the fundamental one of our problem. It corresponds to the equation of p. 43 in our former work.

This equation has the same critical points as that just referred to, viz. 1, $-1$, and $\infty$. Its solution in the neighbourhood of these critical points might be investigated on exactly the same lines as before. The work however would be so nearly identical with our earlier work that it is hardly worth repeating it here.

We shall proceed at once to obtain solutions in a form convenient for physical applications, i.e. in a series of powers of $x$.

Assuming $z = a_0 x^m + a_1 x^{m+1} + \ldots$ and substituting in (2) of last page we find that we must have $m(m-1) = 0$. Thus we get two independent serial solutions, one even and the other odd.

Taking the *even* series first we have:—

$$z = a_0 + a_1 x^2 + \ldots + a_m x^{2m} + \ldots.$$

Substituting in the differential equation and equating coefficients of different powers to zero we get

$$2a_1 + p'a_0 = 0\,;\ (2m+1)(2m+2)\,a_{m+1} + [p' - 2m(2\overline{m+n} - 1)]\,a_m - \lambda^2 a_{m-1} = 0,$$

where $p' = p - n(n+1)$.

If $v_{m+1} = a_{m+1}/a_m$ we find $v_{m+1} = 1 + \dfrac{4(n-1)m - 2 - p'}{(2m+2)(2m+1)} + \dfrac{\lambda^2}{(2m+2)(2m+1)\,v_m}$.

Thus when $m$ *is large, either* $v_m$ is small in which case $v_m$ tends to the limit $-\dfrac{\lambda^2}{(2m+1)(2m+2)}$, *or* $v_m$ is not small and then $v_m$ approaches the limit 1, when $m$ is very large. In the former case the ultimate convergence is the same as that of the series for $\cos \lambda x$, and the series is convergent for all *finite* values of the argument.

We must consider more particularly the other case which is more unfavourable to the convergence of our series. In the first place we note that the series converges for all values of the argument whose modulus is less than unity. It is important to examine the convergence in the extreme case when $|x| = 1$.

The series is $a_0 + a_1 + \ldots + a_m + \ldots$ to $\infty$,

where $$(2m+1)(2m+2)\,a_{m+1} = [2m(2m+2n+1) - p']\,a_m + \lambda^2 a_{m-1}.$$

As we have already seen $\operatorname*{Lt}_{m=\infty} \dfrac{a_{m+1}}{a_m} = 1$,

and when $m$ is very large we have $\dfrac{a_{m+1}}{a_m} = \dfrac{2m(2m+2n+1) - p'}{(2m+1)(2m+2)}$ approx.;

$$\therefore\ m\left(\frac{a_m}{a_{m+1}} - 1\right) = m\,.\,\frac{(2m+1)(2m+2) - 2m(2m+2n+1) + p'}{2m(2m+2n+1) - p'}$$

$$= 1 - n \text{ in the limit when } m = \infty.$$

For all cases except $n = 0$, this is less than 1, so that this test of convergence shows that all the series (corresponding to different values of $n$, except $n = 0$) *diverge* when $x = 1$.

For the case of $n = 0$ we must use a higher test of convergence.

We have in this case:—

$$(\log m) \times \left[m\left(\frac{a_m}{a_{m+1}} - 1\right) - 1\right] = (\log m)\,.\,\frac{p'\,.\,(m+1)}{4m^2 + 2m - p'}$$

$$= \frac{p'}{4}\,.\,\frac{\log m}{m} \text{ ultimately} = \frac{p'}{4}\,.\,\frac{1}{m} \text{ ultimately} = 0,$$

so that for this case also the series *diverges* when $x = 1$.

Now $x'=1$ on the axis of the spheroid, so that if we are dealing with physical problems that require the dependent variable to be expressed in terms of $x$ and $x'$ throughout a region from which the axis of the spheroid is *not* excluded, we cannot make use of the series just considered. For such problems we are confined to the first case mentioned on p. 81. In other words we must have $v_\infty=0$, and then the series we obtain converge for all finite values of the argument. As before this limits us to particular values of $p$.

The series just considered are the *even* series: but of course precisely similar results are obtained from a treatment of the *odd* series.

The cases that will prove most useful to us in the discussion of physical problems are those corresponding to $n=0$ and $n=1$, and we shall therefore consider these more in detail.

For the case of symmetry $n=0$ we have $\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)y=0$. It will be rather more convenient to solve this in powers of $\lambda x$, and taking

$$y=a_0+a_1(\lambda x)^2+\ldots+a_n(\lambda x)^{2n}+\ldots$$

we get

$$2a_1\lambda^2+pa_0=0;\ 12a_2\lambda^2-(6-p)a_1-a_0=0,$$

$$(2n+1)(2n+2)a_{n+1}\lambda^2-[2n(2n+1)-p]a_n-a_{n-1}=0.$$

We have seen that for most physical applications $p$ is confined to the roots of $a_\infty=0$. The determination of these roots proceeds exactly as on p. 60, et seq., the only difference is that in the present case we take $v_n=2n(2n+1)-p$ instead of $(2n)^2-p^2$. Thus we can use our former work to give us the values of $p$ with comparatively very little trouble.

$p_0$ (*corresponding to* $v_0=0$), see p. 60.

*First approximation:* $v_0=v$, $\therefore\ p_0=0$.

*Second.*
$$v_0+\frac{1\,.\,2}{v_1}\lambda^2=0,\ \therefore\ v_0=-\frac{\lambda^2}{3};\ p_0=\frac{\lambda^2}{3}.$$

*Third.*
$$0=v_0+\frac{2\lambda^2}{v_1}-\frac{2\,.\,3\,.\,4}{v_1^2v_2}\lambda^4,$$

giving
$$p_0=\frac{\lambda^2}{3}-\frac{2}{135}\lambda^4.$$

*Fourth.*
$$0=v_0+\frac{2\lambda^2}{v_1}-\frac{2\lambda^4}{v_1}\cdot\frac{3\,.\,4}{v_1v_2}+\frac{2\lambda^6}{v_1}\cdot\frac{3\,.\,4}{v_1v_2}\left(\frac{3\,.\,4}{v_1v_2}+\frac{5\,.\,6}{v_2v_3}\right),$$

giving
$$p_0=\frac{\lambda^2}{3}-\frac{2}{135}\lambda^4+\frac{4}{3^5\,.\,5\,.\,7}\lambda^6,$$

*Fifth.*
$$0=v_0+\frac{2\lambda^2}{v_1}-\frac{2\lambda^4}{v_1}\cdot\frac{3\,.\,4}{v_1v_2}+\frac{2\lambda^6}{v_1}\left(\frac{3\,.\,4}{v_1v_2}\right)\left(\frac{3\,.\,4}{v_1v_2}+\frac{5\,.\,6}{v_1v_2}\right)$$
$$-\frac{2\lambda^8}{v_1}\cdot\frac{3\,.\,4}{v_1v_2}\left[\frac{3\,.\,4}{v_1v_2}\left(\frac{3\,.\,4}{v_1v_2}+\frac{5\,.\,6}{v_2v_3}\right)+\frac{5\,.\,6}{v_2v_3}\left(\frac{3\,.\,4}{v_1v_2}+\frac{5\,.\,6}{v_2v_3}+\frac{7\,.\,8}{v_3v_4}\right)\right].$$

Hence up to $\lambda^{10}$ we have $p_0 = \frac{\lambda^2}{3} - \frac{2}{135}\lambda^4 + \frac{4}{3^5 . 5 . 7}\lambda^6 + \frac{182\lambda^8}{3^7 . 5^3 . 7^3} \dots$

$p_2$ (*corresponding to* $v_1 = 0$) see p. 61.

*First approximation:* $v_1 = 0, \quad \therefore p_2 = 6.$

*Second.* $$v_1 + \lambda^2\left(\frac{1 . 2}{v_0} + \frac{3 . 4}{v_2}\right) = 0; \; p_2 = 6 + \frac{11}{21}\lambda^2.$$

*Third.* $$v_1 + \lambda^2\left(\frac{1 . 2}{v_0} + \frac{3 . 4}{v_2}\right) - \lambda^4\left(\frac{3 . 4}{v_2} . \frac{5 . 6}{v_2 v_3}\right) = 0,$$

$$\therefore p_2 = 6 + \frac{11}{21}\lambda^2 + \frac{94}{(3 \times 7)^3}\lambda^4.$$

*Fourth.* $$v_1 + \lambda^2\left(\frac{1 . 2}{v_0} + \frac{3 . 4}{v_2}\right) - \lambda^4 \frac{3 . 4 . 5 . 6}{v_2^2 v_3} + \lambda^6 \frac{3 . 4 . 5 . 6}{v_2^2 v_3}\left(\frac{5 . 6}{v_2 v_3} + \frac{7 . 8}{v_3 v_4}\right).$$

So that $$p_2 = 6 + \frac{11}{21}\lambda^2 + \frac{94}{(21)^3}\lambda^4 - \frac{21388}{99 \times 7^5}\lambda^6 \dots.$$

$p_4$ (*corresponding to* $v_2 = 0$).

*First approximation:* $v_2 = 0, \quad \therefore p_4 = 20.$

*Second.* $$v_2 + \lambda^2\left(\frac{3 . 4}{v_1} + \frac{5 . 6}{v_3}\right) = 0, \quad \therefore p_4 = 20 + \frac{39}{77}\lambda^2.$$

*Third.* $$v_2 + \lambda^2\left(\frac{3 . 4}{v_1} + \frac{5 . 6}{v_3}\right) - \lambda^4\left[\frac{1 . 2 . 3 . 4}{v_0 v_1^2} + \frac{5 . 6 . 7 . 8}{v_3^2 v_4}\right] = 0,$$

$$\therefore p_4 = 20 + \frac{39}{77}\lambda^2 + \frac{52644}{5 . 7^3 . 11^3 . 13}\lambda^4 + \dots.$$

Similarly with the odd series let

$$y = c_0\lambda x + c_1(\lambda x)^3 + \dots + c_n(\lambda x)^{2n+1} + \dots$$

and we get
$$\begin{aligned}(1 - x^2)&[6c_1\lambda^3 x + 20c_2\lambda^5 x^3 + \dots + 2n(2n+1)c_n\lambda^{2n+1}x^{2n-1} + \dots]\\ &- 2x[c_0\lambda + 3c_1\lambda^3 x^2 + 5c_2\lambda^5 x^4 + \dots + (2n+1)c_n\lambda^{2n+1}x^{2n} + \dots]\\ &+ (p - \lambda^2 x^2)[c_0\lambda x + c_1(\lambda x)^3 + \dots + c_n(\lambda x)^{n+1} + \dots] = 0,\end{aligned}$$

$$\therefore -6c_1\lambda^3 + (2 - p)c_0 = 0; \; -20c_2\lambda^2 + (12 - p)c_1 + c_0 = 0$$

$$-(2n+2)(2n+3)\lambda^2 c_{n+1} + [(2n+1)(2n+2) - p]c_n + c_{n-1} = 0.$$

In this case we take $v_n = (2n+1)(2n+2) - p$ and proceed on exactly the same lines as before.

$p_1$ (*corresponding to root* $v_0 = 0$).

*First approximation:* $v_0 = 0, \quad \therefore p_1 = 2,$

*Second.* $$v_0 + \lambda^2\frac{2 . 3}{v_1} = 0, \quad \therefore p_1 = 2 + \frac{3}{5}\lambda^2.$$

*Third.* $$v_0+\lambda^2\frac{2.3}{v_1}-\lambda^4\frac{2.3.4.5}{v_1^2v_2}=0.$$

Thus $$p_1=2+\frac{3}{5}\lambda^2-\frac{12}{7\times 250}\lambda^4+\dots.$$

$p_3$ (*corresponding to* $v_1=0$).

*First approximation:* $v_1=0,\quad \therefore\ p_3=12.$

*Second.* $$v_1+\lambda^2\left(\frac{2.3}{v_0}+\frac{4.5}{v_2}\right)=0,\quad \therefore\ p_3=12+\frac{23}{45}\lambda^2.$$

*Third.* $$v_1+\lambda^2\left(\frac{2.3}{v_0}+\frac{4.5}{v_2}\right)-\lambda^4\frac{4.5.6.7}{v_2^2v_3}=0,$$

$$\therefore\ p_3=12+\frac{23}{45}\lambda^2+\frac{7229}{2.3^6.5^3.11}\lambda^4+\dots.$$

$p_5$ (*corresponding to* $v_2=0$).

*First approximation:* $v_2=0,\quad \therefore\ p_5=30.$

*Second.* $$v_2+\lambda^2\left(\frac{4.5}{v_1}+\frac{6.7}{v_3}\right)=0,\quad \therefore\ p_5=30+\frac{59}{117}\lambda^2.$$

*Third.* $$v_2+\lambda^2\left(\frac{4.5}{v_1}+\frac{6.7}{v_3}\right)-\lambda^4\left(\frac{2.3.4.5}{v_0v_1^2}+\frac{6.7.8.9}{v_3^2v_4}\right)=0,$$

giving $$p_5=30+\frac{59}{117}\lambda^2+\frac{63338}{3^5.5.7.13^3}\lambda^4+\dots.$$

Turning now from the case of symmetry ($n=0$) to the case $n=1$, our equation becomes $(1-x^2)\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+(p-2-\lambda^2x^2)\,y=0.$

If $$y=a_0+a_1(\lambda x)^2+\dots+a_n(\lambda x)^{2n}+\dots$$

we have $$2a_1\lambda^2+(p-2)\,a_0=0;\quad 12a_2\lambda^2+(p-12)\,a_1-a_0=0,$$

$$-(2n+1)(2n+2)\,a_{n+1}\lambda^2+[(2n+1)(2n+2)-p]\,a_n+a_{n-1}=0,$$

and in the determination of $p$ we proceed as before, the only difference being that now we have $v_n=(2n+1)(2n+2)-p$. In this way we get

$$p_0=2+\frac{1}{5}\lambda^2-\frac{4}{7.5^3}\lambda^4+\frac{8}{21\times 5^5}\lambda^6+\dots,$$

$$p_2=12+\frac{7}{15}\lambda^2+\frac{1064}{77.3^4.5^3}\lambda^4+\dots,$$

$$p_4=30+\frac{19}{39}\lambda^2+\frac{1108}{7.3^4.13^3}\lambda^4+\dots$$

For the *odd* series $y = c_0\lambda x + c_1(\lambda x)^3 + \ldots + c_n(\lambda x)^{2n+1} + \ldots,$

we have
$$6c_1\lambda^2 + (p-6)\,c_0 = 0\,; \quad 20c_2\lambda^2 + (p-20)\,c_1 - c_0 = 0$$
$$-(2n+2)(2n+3)\,c_{n+1}\lambda^2 + [(2n+2)(2n+3) - p]\,c_n + c_{n-1} = 0,$$

giving by the usual method
$$p_1 = 6 + \frac{3}{7}\lambda^2 - \frac{4}{1029}\lambda^4 + \frac{8}{33.7^5}\lambda^6 + \ldots,$$
$$p_3 = 20 + \frac{37}{77}\lambda^2 + \frac{7064}{7^3.11^2.13}\lambda^4 + \ldots,$$
$$p_5 = 42 + \frac{27}{55}\lambda^2 + \ldots.$$

Just as on p. 63 the odd and even series obtained as solutions of our differential equation correspond to different values of $p$ and so are really solutions of different equations. But having obtained one solution of our equation there is no difficulty in finding another. In the case of symmetry our equation is
$$\frac{d}{dx}(1-x^2)\frac{dy}{dx} + (p - \lambda^2x^2)\,y = 0.$$

Take $y = uf + w$ where $f$ is the solution already found.

We then have
$$\frac{d}{dx}(1-x^2)\frac{dw}{dx} + (p-\lambda^2x^2)\,w + f.\frac{d}{dx}(1-x^2)\frac{du}{dx} + 2(1-x^2)\frac{du}{dx}\frac{df}{dx} = 0.$$

Choose $u$ so that $\frac{d}{dx}(1-x^2)\frac{du}{dx} = 0$, which is satisfied by $u = \frac{1}{2}\log\frac{x+1}{x-1}$.

Our equation then reduces to $\frac{d}{dx}(1-x^2)\frac{dw}{dx} + (p-\lambda^2x^2)\,w = -2\frac{df}{dx}$, and we want a particular value of $w$ (the simpler the better) to satisfy this.

If $f = a_0 + a_1x^2 + \ldots$ (even), we assume $w = A_0x + A_1x^3 + \ldots$ (odd).

Substituting in the differential equation and equating coefficients we get
$$6A_1 + (p-2)\,A_0 = -4a_1; \quad 20A_2 + (p-12)\,A_1 - \lambda^2A_0 = -8a_2,$$
$$(2n+2)(2n+3)\,A_{n+1} + [p-(2n+1)(2n+2)]\,A_n - \lambda^2A_{n-1} = -4(n+1)\,a_{n+1},$$

from which we can determine the coefficients, in succession $A_0$ (or $A_1$) being chosen at pleasure.

With the aid of these relations we can show just as on p. 65 that the series for $w$ is convergent in the same domain as $f$.

Thus we have two independent solutions of our equation
$$y = f(x) \text{ and } y = \tfrac{1}{2}\log\frac{x+1}{x-1} \times f(x) + w = F(x),$$

the complete integral being $y = Af(x) + BF(x)$.

The series defining $f$ is convergent for all finite values of the argument; but $F$ becomes infinite when $x=\pm 1$, and so must not be used in regions where $x$ can have either of these values.

We proceed to find some series that approach 'asymptotically' to solutions of our equation and that will prove useful for numerical calculation for large values of the argument.

Putting $z=i\lambda x$ and $y=ue^z$, the equation

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)\,y=0 \text{ becomes } (z^2+\lambda^2)\frac{d^2u}{dz^2}+2(z^2+z+\lambda^2)\frac{du}{dz}+(2z-p+\lambda^2)\,u=0.$$

Assume $u=\frac{b_0}{z}+\frac{b_1}{z^3}+\dots\frac{b_n}{z^{n+1}}+\dots$ and we get

$$(z^2+\lambda^2)\left[\frac{2b_0}{z^3}+\frac{6b_1}{z^4}+\frac{12b_2}{z^5}+\dots+n(n+1)\frac{b_{n-1}}{z^{n+2}}+\dots\right]$$
$$-2(z^2+z+\lambda^2)\left[\frac{b_0}{z^2}+\frac{2b_1}{z^3}+\dots+\frac{b_{n-1}}{z^{n+1}}+\dots\right]$$
$$+(2z-p+\lambda^2)\left[\frac{b_0}{z}+\frac{b_1}{z^2}+\dots+\frac{b_n}{z^{n+1}}+\dots\right]=0.$$

Hence $\quad +2b_1+(p-\lambda^2)\,b_0=0;\quad 4b_2+(p-2-\lambda^2)\,b_1+\lambda^2b_0=0,$

$$2nb_n=[n(n-1)+\lambda^2-p]\,b_{n-1}-\lambda^2(n-1)\,[2b_{n-2}-\overline{n-2}\,b_{n-3}].$$

So that if $u_n=b_n/b_{n-1}$ we have

$$2u_n=n-1+\frac{\lambda^2-p}{n}-\lambda^2\frac{n-1}{n}\left[\frac{2}{u_{n-1}}-(n-2)\frac{1}{u_{n-1}u_{n-2}}\right].$$

Thus when $n$ is large, either $u_n$ is large and equal to $\frac{n-1}{2}$ approximately, or if $u_n$ is small we have when $n$ is large $0=1-\frac{\lambda^2}{n}\left[\frac{2}{u_{n-1}}-\frac{n-2}{u_n\,.\,u_{n-1}}\right]$, and as this cannot be satisfied by a small value of $u_n$, we see that $u_n$ is ultimately large, so that the series is ultimately divergent. But as before it is useful for arithmetical purposes if we confine ourselves to the convergent part, the argument being large.

The solution thus obtained is (if $z=\lambda x$),

$$y=\frac{e^{iz}}{z}\left[b_0+\frac{b_1}{iz}+\frac{b_2}{(iz)^2}+\dots\right]$$
$$=\frac{e^{iz}}{z}\left[b_0-i\frac{b_1}{z}-\frac{b_2}{z^2}+\dots\right]=\phi^+(z) \text{ say.}$$

Changing the sign of $i$ we get another solution

$$y=\frac{e^{-iz}}{z}\left[b_0+i\frac{b_1}{z}-\frac{b_2}{z^2}+\dots\right]=\phi^-(z),$$

and the complete integral is $y=C\phi^+(z)+D\phi^-(z)$ where $C$ and $D$ are arbitrary constants.

As on p. 67 we shall take $\chi^+(z)=\dfrac{\phi^+(z)+\phi^-(z)}{2}$; $\chi^-(z)=i.\dfrac{\phi^-(z)-\phi^+(z)}{2}$,

giving
$$\chi^+(z)=\frac{1}{z}[R\cos z+S\sin z],$$
$$\chi^-(z)=\frac{1}{z}[R\sin z-S\cos z],$$
where
$$R=b_0-\frac{b_2}{z^2}+\ldots;\ S=\frac{b_1}{z}-\frac{b_3}{z^3}+\ldots.$$

By proceeding very much as on p. 68 we can obtain some relations between the various solutions of our equation. Take $z=r(\cos\theta+i\sin\theta)$, obviously we have

$\phi^+(-z)=-\phi^-(z)$; $\phi^-(-z)=-\phi^+(z)$ or $\phi^+(\theta+\pi)=-\phi^-(\theta)$; $\phi^-(\theta+\pi)=-\phi^+(\theta)$.

Also we have $F(z)=\frac{1}{2}\log\dfrac{z+\lambda}{z-\lambda}f(z)+w(z)$ where $w(z)$ is odd if $f$ is even and even if $f$ is odd.

Taking $f$ even, we have $F(-z)=\frac{1}{2}\log\dfrac{-z+\lambda}{-z-\lambda}f(-z)+w(-z)$
$$=-\left[\tfrac{1}{2}\log\frac{z+\lambda}{z-\lambda}f(z)+w(z)\right]=-F(z),$$
and if $f$ is odd we get similarly $F(-z)=+F(z)$.

Then proceeding as on p. 69 we find these relations:—

*if $f$ is even,*
$$f=\alpha'(\phi^+-\phi^-)=\alpha\chi^-,$$
$$F=\beta'(\phi^++\phi^-)=\beta\chi^+,$$
and *if $f$ is odd,*
$$f=\alpha_1'(\phi^++\phi^-)=\alpha_1\chi^+,$$
$$F=\beta_1'(\phi^+-\phi^-)=\beta_1\chi^-.$$

The larger roots of $f(z)$ can be got from these results by the same process as on p. 70.

For the even series $(f_0f_2\ldots)$ we require the roots of $\chi^-(z)=0$,

i.e.
$$R\sin z-S\cos z=0.$$

As before, putting
$$\tan\psi=\frac{S}{R}=\frac{\dfrac{b_1}{z}-\dfrac{b_3}{z^3}+\ldots}{b_0-\dfrac{b_2}{z^2}+\ldots},$$
our equation becomes $\tan(z-\psi)=0$, so that $z=m\pi+\psi$ where $m$ is an integer.

Similarly with the odd series; and with the roots of $f'(z)=0$. The smaller roots may be got by Horner's method as already illustrated on p. 72 directly from the

ascending series. For example, I find that the equation to determine the roots of $f_1'(z)=0$ is

$$\begin{aligned}0 &= 1 - \cdot 3z^2 + (\cdot 017857 + \cdot 0034285e^2)\,z^4 \\ &\quad - (\cdot 00059524 + \cdot 00031747e^2 - \cdot 00003047e^4)\,z^6 \\ &\quad + (\cdot 00006764 + \cdot 0000111e^2 - \cdot 0000010885e^4)\,z^8 + \ldots,\end{aligned}$$

and the lowest root for $e=0$, $e=0{\cdot}1$; $e=0{\cdot}2$; $e=0{\cdot}3\ldots$, $e=0{\cdot}9$ is 2·0815; 2·0825; 2·0848; 2·0865; 2·0902; 2·0961; 2·1035; 2·1124; 2·1233; 2·1364.

The treatment of the equation corresponding to $n=1$ instead of $n=0$ of our fundamental equation proceeds on quite similar lines. We have already seen that on putting $y=\sqrt{1-x^2}\,.\,y_1$, the equation becomes $(1-x^2)\dfrac{d^2y_1}{dx^2}-4x\dfrac{dy_1}{dx}+(p-2-\lambda^2x^2)\,y_1=0$ and we have obtained solutions of this in the form of *ascending* series of $x$. To obtain a solution corresponding to that on p. 86 we put $z=i\lambda x$ and $y_1=ue^z$, and the equation becomes

$$(z^2+\lambda^2)\frac{d^2u}{dz^2}+2\,[z^2+\lambda^2+2z]\frac{du}{dz}+(4z-p+2+\lambda^2)\,u=0.$$

Assume
$$u=\frac{b_0}{z^2}+\frac{b_1}{z^3}+\frac{b_2}{z^4}+\ldots+\frac{b_n}{z^{n+2}}+\ldots,$$

and we get $2b_1=(\lambda^2-p)\,b_0$; $4b_2=(\lambda^2+2-p)\,b_1-4\lambda^2b_0$

$$2\,(n+1)\,b_{n+1}=[\lambda^2+n\,(n+1)-p]\,b_n+(n+1)\,\lambda^2\,[nb_{n-2}-2b_{n-1}].$$

As before the series is ultimately divergent, but is none the less useful for purposes of calculation. The work on p. 87 is, with very slight and obvious modifications, applicable to these functions.

If $y$ and $y_1$ are two solutions of our fundamental equation corresponding to different values of $\kappa$, then $y$ and $y_1$ have a "*conjugate*" *property*, which will be of use to us.

We have
$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+(p-\lambda^2x^2)\,y=0,$$

$$\frac{d}{dx}(1-x^2)\frac{dy_1}{dx}+(p_1-\lambda_1^2x^2)\,y_1=0.$$

Hence
$$\int_{-1}^{1}[p-p_1-(\lambda^2-\lambda_1^2)\,x^2]\,yy_1\,dx=\int_{-1}^{1}\left[y\frac{d}{dx}(1-x^2)\frac{dy_1}{dx}-y_1\frac{d}{dx}(1-x^2)\frac{dy}{dx}\right]dx$$
$$=\left[y\,(1-x^2)\frac{dy_1}{dx}-y_1(1-x^2)\frac{dy}{dx}\right]_{-1}^{1}=0.$$

By aid of this property we can determine the coefficients in the expansion of any function of $x'$ in a series of terms like $f_n(x')$ by a process similar to that used when dealing with Legendre's or Bessel's functions.

In the case where $\lambda=\lambda_1$ but $p\neq p_1$ the conjugate property takes the simpler form $\int_{-1}^{1}yy_1\,dx=0$.

The expansion of the function $e^{i\lambda xx'}$ in terms of our functions is interesting in itself and leads to some results of importance.

If $x_1$ be the ordinary Cartesian coordinate of a point (the axis of $x_1$ being the axis of the spheroid) we have $x_1 = hxx'$ in our notation. Now $e^{ixx_1}$ obviously satisfies $(V^2 + \kappa^2)\psi = 0$, and is finite when $x = \pm 1$, so that it must be expressible in a series of $f$ functions.

Assume then $$e^{i\lambda xx'} \equiv \Sigma A_n f_n(\lambda x) f_n(\lambda x').$$

By p. 88 we have in this case

$$\int_{-1}^{1} f_n(\lambda x') f_m(\lambda x')\, dx' = 0.$$

Hence $$\int_{-1}^{1} e^{i\lambda xx'} f_n(\lambda x')\, dx' \equiv A_n f_n(\lambda x) \int_{-1}^{1} \overline{f_n(\lambda x')}|^2\, dx',$$

a relation which must hold for all values of $\lambda x$, $(= z$ say).

Now $$e^{i\lambda xx'} = e^{izx'} = 1 + izx' - \frac{z^2x'^2}{\underline{|2}} - \frac{iz^3x'^3}{\underline{|3}} + \dots$$

and $$f_n(z) = a_0 + a_1 z^2 + \dots \text{ if } n \text{ is even,}$$

and $$= c_0 z + c_1 z^3 + \dots \text{ if } n \text{ is odd.}$$

Hence, if $n$ *is even*,

$$\int_{-1}^{1} \left(1 + izx' - \frac{z^2x'^2}{\underline{|2}} - \frac{iz^3x'^3}{\underline{|3}} + \dots\right) f_n(\lambda x')\, dx'$$
$$\equiv \left\{A_n \int_{-1}^{1} \overline{f_n(\lambda x')}|^2\, dx'\right\} (a_0 + a_1 z^2 + a_2 z^4 + \dots).$$

Equating the coefficients of different powers of $z$ we see that $\int_{-1}^{1} x'^{2r+1} f_n(\lambda x')\, dx' = 0$, as is otherwise evident since $f_n(\lambda x')$ is an *even* function.

We also get $$\int_{-1}^{1} f_n(\lambda x')\, dx' = a_0 A_n \int_{-1}^{1} \overline{f_n(\lambda x')}|^2\, dx',$$

which determines $A_n$, see p. 90,

and $$-\frac{1}{\underline{|2}} \int_{-1}^{1} x'^2 f_n(\lambda x')\, dx' = \frac{a_1}{a_0} \int_{-1}^{1} f_n(\lambda x')\, dx',$$

$$\frac{1}{\underline{|4}} \int_{-1}^{1} x'^4 f_n(\lambda x')\, dx' = \frac{a_2}{a_0} \int_{-1}^{1} f_n(\lambda x')\, dx',$$

and so on.

Similarly if $n$ *is odd* we have

$$\int_{-1}^{1} \left(1 + izx' - \frac{z^2x'^2}{\underline{|2}} - \frac{iz^3x'^3}{\underline{|3}} + \dots\right) f_n(\lambda x')\, dx'$$
$$\equiv \left\{A_n \int_{-1}^{1} \overline{f_n(\lambda x')}|^2\, dx'\right\} (c_0 z + c_1 z^3 + c_2 z^5 + \dots).$$

This gives $\int_{-1}^{1} x'^{2m} f_n(\lambda x')\,dx'=0$, as is obvious, since $f_n$ is an *odd* function.

Also
$$i\int_{-1}^{1} x' f_n(\lambda x')\,dx' = c_0 A_n \int_{-1}^{1} \overline{f_n(\lambda x')}|^2 dx',$$

which determines $A_n$, see below.

And
$$-\frac{1}{\underline{|3}}\int_{-1}^{1} x'^3 f_n(\lambda x')\,dx' = \frac{c_1}{c_0}\int_{-1}^{1} x' f_n(\lambda x')\,dx',$$
$$\frac{1}{\underline{|5}}\int_{-1}^{1} x'^5 f_n(\lambda x')\,dx' = \frac{c_2}{c_0}\int_{-1}^{1} x' f_n(\lambda x')\,dx',$$
$$\frac{(-1)^m}{\underline{|2m+1}}\int_{-1}^{1} x'^{2m+1} f_n(\lambda x')\,dx' = \frac{c_m}{c_0}\int_{-1}^{1} x' f_n(\lambda x')\,dx'.$$

Now when $n$ is even $f_n(\lambda x)=a_0+\ldots a_1(\lambda x)^2+a_2(\lambda x)^4+\ldots$,
$$\therefore \int_{-1}^{1} \overline{f_n(\lambda x)}|^2 dx = a_0\int_{-1}^{1} f_n(\lambda x)\,dx + a_1\lambda^2\int_{-1}^{1} x^2 f_n(\lambda x)\,dx+\ldots$$
$$=\frac{1}{a_0}[a_0^2-a_1^2\lambda^2\,\underline{|2}+a_2^2\lambda^4\,\underline{|4}\ldots]\int_{-1}^{1} f_n(\lambda x)\,dx,$$

by p. 89.

Similarly, when $n$ is odd, we have $f_n(\lambda x)=c_0\lambda x+c_1(\lambda x)^3+\ldots$,
$$\therefore \int_{-1}^{1} \overline{f_n(\lambda x)}|^2 dx = \frac{1}{c_0}[c_0^2\lambda - c_1^2\lambda^3\,\underline{|3}+c_2^2\lambda^5\,\underline{|5}+\ldots]\int_{-1}^{1} x f_n(\lambda x)\,dx,$$

by means of the relations above.

Since $\int_{-1}^{1} f_n(\lambda x)\,dx$ and $\int_{-1}^{1} x f_n(\lambda x)\,dx$ are easily determined by direct integration of the series, we can thus determine $\int_{-1}^{1} \overline{f_n(\lambda x)}|^2\,dx$.

The above gives us the value of the constant $A_n$ of this and the last page. When $n$ is even we have
$$A_n=\frac{\int_{-1}^{1} f_n(\lambda x)\,dx}{a_0\int_{-1}^{1} \overline{f_n(\lambda x)}|^2\,dx},$$

so that in this case $\qquad A_n=1/(a_0^2-a_1^2\lambda^2\,\underline{|2}+a_2^2\lambda^4\,\underline{|4}+\ldots)$,

and similarly, when $n$ is odd, we get
$$A_n=i/c_0^2\lambda^2-c_1^2\lambda^3\,\underline{|3}+c_2^2\lambda^5\,\underline{|5}+\ldots).$$

We shall proceed to discuss some physical problems, involving the use of the functions now under consideration.

*Vibrations of spheroidal sheets of air.*

If $c$ be the velocity of sound in air and $\psi$ the velocity potential our characteristic equation is $\frac{\partial^2\psi}{\partial t^2} = c^2 V^2\psi$. Taking $\psi$ to vary as $e^{i\kappa ct}$ we get $(V^2+\kappa^2)\psi = 0$. As we are confined to a thin layer we may regard $x$ as a constant, and so our fundamental equation reduces to $\frac{d}{dx'}(1-x'^2)\frac{d\psi}{dx'} + (\kappa^2a^2 - \lambda^2x'^2)\psi = 0$. This is the same form as before with $p = \kappa^2a^2$.

If the layer is *complete* the series for $\psi$ must be convergent for all possible values of $x'$ (i.e. from $x' = -1$ to $x' = 1$, inclusive), and so we have $\psi = Af(x')$; [$F(x')$ becomes infinite at the poles $x' = \pm 1$] and the necessity of convergence at the poles determines the hitherto arbitrary constant $\kappa$, or rather confines it to certain definite values. See p. 82.

If the layer of air instead of being complete is bounded by two parallels of latitude, then our solution is $\psi = Af_0(x') + Bf_1(x')$ where $f_0$ and $f_1$ are the odd and even series of p. 81. $p$ is not now restricted by conditions of convergence, the series $f(x')$ being convergent (whatever $p$ be) provided $x' < 1$, which is the case in the present problem.

At the bounding walls we must have $\frac{d\psi}{dx'} = 0$; so that if $x' = \alpha$, $x' = \beta$ are the boundaries, we have

$$Af_0'(\alpha) + Bf_1'(\alpha) = 0,$$
$$Af_0'(\beta) + \beta f_1'(\beta) = 0.$$

Eliminating $A:B$ we get the frequency equation

$$\begin{vmatrix} f_0'(\alpha), & f_1'(\alpha) \\ f_0'(\beta), & f_1'(\beta) \end{vmatrix} = 0,$$

which determines the admissible values of $\kappa$.

If one of the boundaries ($\beta$) is the equator, then the frequency equation takes the simple form $f_0'(\alpha) = 0$.

Returning to the case of the complete spheroid we shall determine the frequency corresponding to some of the simpler modes of vibration.

*Corresponding to* $p_0$: (p. 82),

we have
$$\kappa^2a^2 = p_0 = \frac{1}{3}\lambda^2 - \frac{2}{135}\lambda^4 + \frac{4}{3^5 . 5 . 7}\lambda^6 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and $\lambda = h\kappa = \kappa ae$, so that (1) becomes

$$\kappa^2a^2\left[1 - \frac{1}{3}e^2 + \frac{2}{135}\kappa^2a^2e^4 + \ldots\right] = 0.$$

It follows from this that for our problem we must have $\kappa a = 0$. The expression in brackets [ ] cannot vanish for real values of $\kappa a$, to which we are confined in the present case.

$\kappa$ being zero, $\psi$ is independent of the time—so that there is really no *vibration* at all, and this so called 'mode' is without physical interest.

*Corresponding to* $p_1$. This is the *slowest* periodic movement for the case of symmetry round the axis. The vibration is *un*symmetrical with respect to the ends of the spheroid.

We have
$$\kappa^2a^2 = p_1 = 2 + \frac{3}{5}\lambda^2 - \frac{6}{875}\lambda^4 - \frac{4}{3\,.\,5^5\,.\,7}\lambda^6 + \dots$$
$$= 2 + \frac{3}{5}\kappa^2a^2e^2 - \frac{6}{875}\kappa^4a^4e^4 + \dots,$$
giving
$$\kappa^2a^2 = 2 + 1\,.\,2e^2 + \cdot 692e^4 + \cdot 38e^6 + \dots.$$

| $e$ | $\kappa^2a^2$ | $a \times$ frequency | wave length/$a$ |
|---|---|---|---|
| 0 | 2 | 7439 | 4·450 |
| 0·1 | 2·0121 | 7461 | 4·436 |
| 0·2 | 2·0491 | 7528 | 4·396 |
| 0·3 | 2·1136 | 7647 | 4·328 |
| 0·4 | 2·209 | 7818 | 4·234 |
| 0·5 | 2·3491 | 8061 | 4·106 |
| 0·6 | 2·5397 | 8383 | 3·948 |
| 0·7 | 2·7892 | 8786 | 3·767 |
| 0·8 | 3·1497 | 9335 | 3·546 |
| 0·9 | 3·6181 | 10,000 | 3·310 |

The ratios of the frequencies for $e=0{\cdot}1 \dots e=0{\cdot}9$ to that for the sphere ($e=0$) are 1·003; 1·012; 1·028; 1·051; 1·083; 1·126; 1·181; 1·214; 1·330.

The interval between successive cases is rather more than a *comma*, that between $e=0{\cdot}8$ and $e=0$ is rather more than a *minor third*, and between $e=0{\cdot}9$ and $e=0$ is just about a *fourth*.

If then the sphere gives out the note $c$ (or *Do*)

the spheroid $e=\frac{6}{10}$ " " " " $d$ (or *Re*) very nearly,

" " $e=\frac{8}{10}$ " " " " $e$ (or *Mi*) fairly nearly,

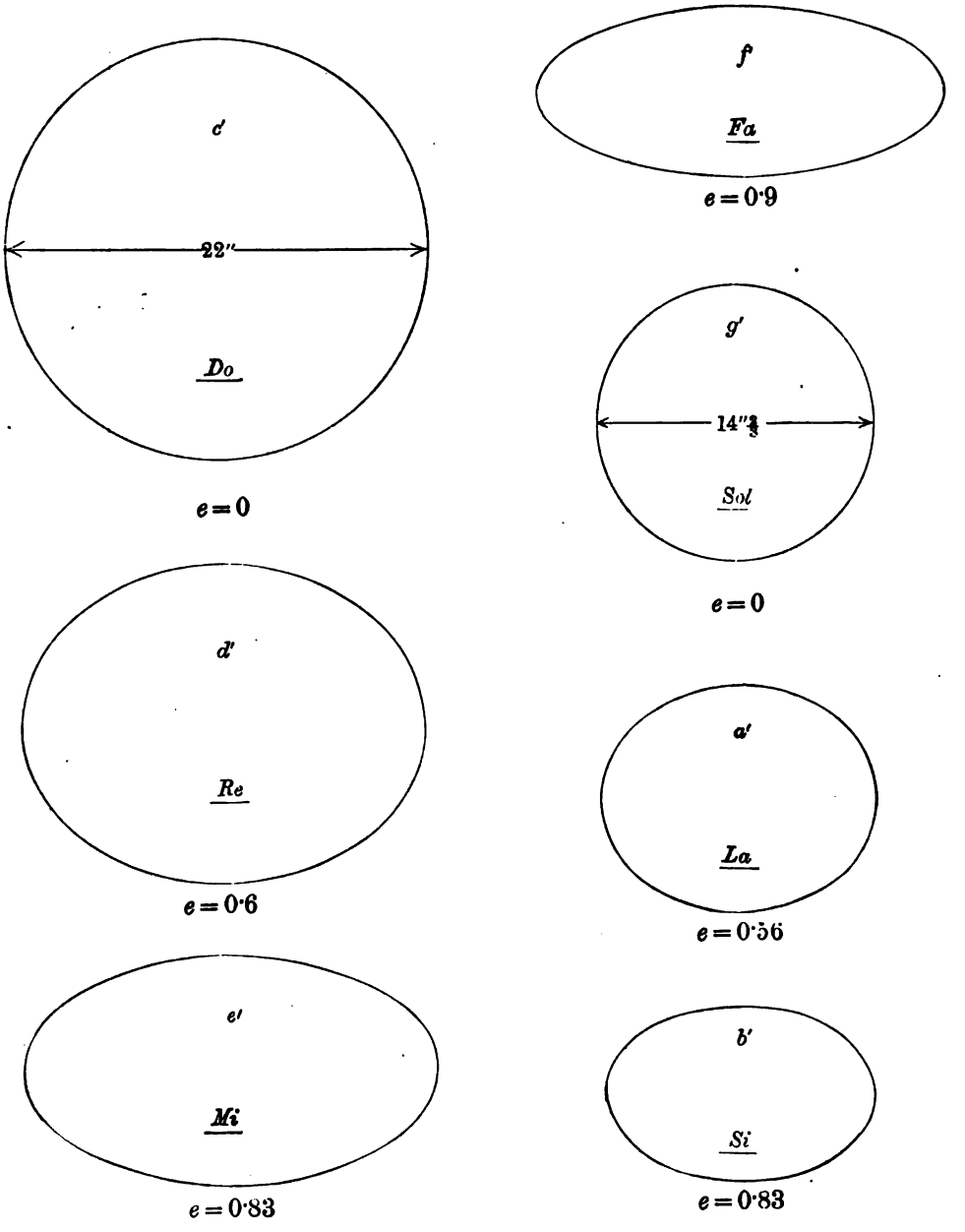
" " $e=\frac{9}{10}$ " " " " $f$ (or *Fa*) very nearly.

For the sphere to sound the middle $c$ of a piano ($c'$; frequency 264) it must have a radius $a = 28$ centimetres $= 11$ inches.

If the eccentricity is $\frac{5}{10}$ then $a = 31$ centimetres ($12\frac{1}{5}$ inches).

" " " $\frac{9}{10}$ " $a = 37$ " ($14\frac{3}{5}$ inches).

The accompanying rough sketch will illustrate to the eye the influence of size and eccentricity on the note emitted.



*Corresponding to* $p_2$ (the slowest *symmetrical* vibration),

$$\kappa^2a^2=p_2=6+\frac{11}{21}\lambda^2+\frac{94}{(21)^2}\lambda^4-\frac{21388}{99\times 7^5}\lambda^6\ldots \quad \text{(p. 83)}$$

$$=6+\frac{11}{21}\kappa^2a^2e^2+\frac{94}{(21)^2}\kappa^4a^4e^4-\frac{21388}{7^5\times 99}\kappa^6a^6e^6+\ldots,$$

so that $$\kappa^2a^2=6+3{\cdot}14e^2+2e^4-1{\cdot}4e^6+\ldots$$

| $e$ | $\kappa^2a^2$ | $a\times$ frequency | wave length/$a$ | Ratio of frequency to that for $e=0$ |
|---|---|---|---|---|
| 0 | 6 | 12883 | 2·569 | 1 |
| 0·1 | 6·0314 | 12918 | 2·563 | 1·003 |
| 0·2 | 6·1286 | 13020 | 2·544 | 1·010 |
| 0·3 | 6·2978 | 13201 | 2·507 | 1·024 |
| 0·4 | 6·5459 | 13461 | 2·459 | 1·045 |
| 0·5 | 6·9136 | 13830 | 2·393 | 1·073 |
| 0·6 | 7·3238 | 14234 | 2·325 | 1·105 |
| 0·7 | 7·8456 | 14734 | 2·247 | 1·143 |
| 0·8 | 8·4620 | 15301 | 2·163 | 1·186 |
| 0·9 | 9·3266 | 16063 | 2·061 | 1·247 |

The interval between the notes corresponding to $e=0$ and $e=\frac{2}{10}$ is just about a *comma:* between those for $e=0$ and $e=\frac{9}{10}$ about a *major third.*

We shall now turn to the case of *non*-symmetry about the axis. Taking $\psi$ to vary as $\sin\phi$ (where $\phi$ is the azimuth) we have $n=1$ in the equation of p. 80.

*Corresponding to* $p_0$ (the gravest note we can get),

$$\kappa^2a^2=p_0=2+\frac{1}{5}\lambda^2-\frac{4}{5^3.7}\lambda^4+\frac{8}{21.5^5}\lambda^6\ldots$$

$$=2+\frac{1}{5}\kappa^2a^2e^2-\frac{4}{5^3.7}(\kappa a)^4e^4+\frac{8}{21.5^5}(\kappa a)^6e^6+\ldots,$$

giving $$\kappa^2a^2=2+0{\cdot}4e^2+0{\cdot}062e^4+0{\cdot}006e^6+\ldots.$$

The values are tabulated on the next page. The lowest note in the series (that for the sphere) is the same as that in the mode discussed on p. 92. But for the same eccentricity (except $e=0$) the notes in this mode are rather lower than in the mode on p. 92. In the present case if the sphere gives the note *Do* (*c*), then the spheroid $e=\frac{9}{10}$ gives out a note slightly *lower* than *Re* (*d*), while the note corresponding

to $e=\frac{9}{10}$ in the mode of p. 92 is $Fa(f)$. The spheroid $e=\frac{5}{10}$ gives in the mode of p. 92 about the same note as $e=\frac{9}{10}$ in the present mode.

| $e$ | $\kappa^2a^2$ | $a\times$ frequency | wave length/$a$ |
|---|---|---|---|
| 0 | 2 | 7439 | 4·450 |
| 0·1 | 2·004 | 7446 | 4·445 |
| 0·2 | 2·016 | 7468 | 4·432 |
| 0·3 | 2·037 | 7508 | 4·409 |
| 0·4 | 2·066 | 7561 | 4·377 |
| 0·5 | 2·104 | 7630 | 4·338 |
| 0·6 | 2·152 | 7716 | 4·289 |
| 0·7 | 2·211 | 7822 | 4·231 |
| 0·8 | 2·282 | 7944 | 4·167 |
| 0·9 | 2·367 | 8093 | 4·090 |

*Corresponding to* $p_1$

$$\kappa^2a^2=p_1=6+\frac{3}{7}\lambda^2-\frac{4}{1029}\lambda^4+\frac{8}{33\,.\,7^5}\lambda^6+\dots$$

$$=6+\frac{3}{7}\kappa^2a^2e^2-\frac{4}{1029}(\kappa a)^4e^4+\frac{8}{33\,.\,7^5}(\kappa a)^6e^6+\dots,$$

so that

$$\kappa^2a^2=6+2{\cdot}57e^2+0{\cdot}96e^4+0{\cdot}33e^6+\dots.$$

| $e$ | $\kappa^2a^2$ | $a\times$ frequency | wave length/$a$ |
|---|---|---|---|
| 0 | 6 | 12883 | 2·569 |
| 0·1 | 6·0257 | 12916 | 2·562 |
| 0·2 | 6·1032 | 12996 | 2·546 |
| 0·3 | 6·2393 | 13141 | 2·519 |
| 0·4 | 6·4375 | 13347 | 2·480 |
| 0·5 | 6·6067 | 13528 | 2·446 |
| 0·6 | 7·0632 | 13980 | 2·366 |
| 0·7 | 7·5226 | 14427 | 2·294 |
| 0·8 | 8·1242 | 14994 | 2·207 |
| 0·9 | 8·8902 | 15686 | 2·110 |

The notes of this mode are rather lower than in the corresponding cases on p. 94; $e=\frac{9}{10}$ of this mode nearly corresponding with $e=\frac{8}{10}$ of the former.

*Vibrations in a hemi-spheroidal sheet.*

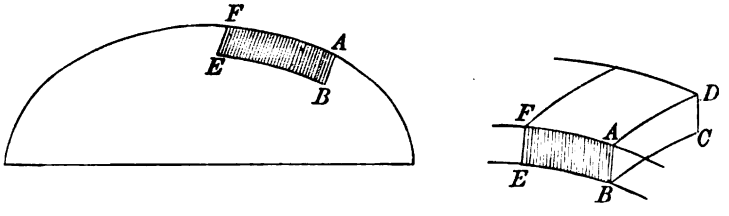
The symmetrical vibrations (corresponding to $p_2$, $p_4 \ldots$) make the equator *nodal*, so that we may at once adopt the above results to vibrations in a hemispheroid *closed at the edge*. The notes are given on p. 94.

The unsymmetrical type (corresponding to $p_1$, $p_3 \ldots$) make $\psi=0$ and therefore $\dot{\psi}=0$ at the equator ($x'=0$). Thus in these modes there is no pressure variation at the equator, so that there is a *loop* there. If we make the rough assumption usually adopted in the elementary treatment of open ends—viz. that there is a loop at the end, we can thus apply our results to the determination of the notes in a hemi-spheroidal layer with an *open end* (open at the equator). The notes are given on pp. 92 and 95.

The case discussed on last page ($n=1$) gives us the gravest note in a *quadrant of a spheroid*, closed at the sides and open at the equator.

A problem that is analytically closely analogous to the one just discussed is the investigation of the modes of vibration for the *electrical oscillations in a thin homœoidal layer of dielectric* between two conducting spheroidal surfaces. There are clearly two perfectly distinct types of oscillation. In one the wave surges backwards and forwards between the bounding surfaces, the *magnetic* force being normal to the ellipsoid and so the electric force tangential. In this type the wave length will be comparable with the thickness $\tau$ of the layer, and the frequency will be consequently excessively great. The longest wave will be that which crosses the layer and gets back again in a period, so that its wave length is $2\tau$ and there will be harmonics of wave lengths $\tau$, $\frac{2}{3}\tau$, etc. For the other type—which we are now to examine—the waves advance along the ellipsoidal surface, the electric force being normal to the ellipsoid and the magnetic force tangential.

The electric force $R$ is normal to the spheroid. Let the line element on the spheroid be given by $ds^2=\frac{dx'^2}{h_1^2}+\frac{d\phi^2}{h_2^2}$.




Then, on applying the fundamental circuital laws of Faraday and Ampère to the elements $ABEF$ and $ABCD$, we get $\left\{\text{with help of relation } w=\frac{K}{4\pi}\frac{\partial R}{\partial t}\right\}$,

$$\mu K\frac{\partial^2 R}{\partial t^2}=\mu\,.\,h_1h_2\left[\frac{\partial}{\partial x'}\left(\frac{h_1}{h_2}\cdot\frac{1}{\mu\tau}\frac{\partial(\tau R)}{\partial x'}\right)+\frac{\partial}{\partial\phi}\left(\frac{h_2}{h_1}\cdot\frac{1}{\mu\tau}\frac{\partial(\tau R)}{\partial\phi}\right)\right].$$

Owing to the symmetry about the axis $\tau R$ is independent of $\phi$ and our equation becomes

$$\mu K \frac{\partial^2 R}{\partial t^2} = \mu h_1 h_2 \frac{\partial}{\partial x'}\left(\frac{h_1}{h_2} \cdot \frac{1}{\mu\tau}\frac{\partial \tau R}{\partial x'}\right)$$

$$= h_1 h_2 \frac{\partial}{\partial x'}\left(\frac{h_1}{h_2}\frac{1}{\tau}\frac{\partial(\tau R)}{\partial x'}\right) \quad \text{.....(1)},$$

if $\mu$ is constant, as we shall suppose.

Now $$h_1^2 = \frac{p'^2}{h^4 x'^2} = \frac{1 - x'^2}{h^2(x^2 - x'^2)}; \quad h_2^2 = \frac{1}{h^2(x^2-1)(1-x'^2)},$$

and for a homœoid $\tau = \alpha p$ where $\alpha$ is a constant, or $\tau = \alpha . \dfrac{hx\sqrt{x^2-1}}{\sqrt{x^2-x'^2}}$.

Substituting in (1) we get

$$\mu K \frac{\partial^2 R}{\partial t^2} = \frac{1}{h^2\sqrt{x^2-x'^2}}\frac{\partial}{\partial x'}(1-x'^2)\frac{\partial}{\partial x'}\frac{R}{\sqrt{x^2-x'^2}};$$

or if $$\psi = R/\sqrt{x^2-x'^2}, \quad \mu K h^2(x^2-x'^2)\frac{\partial^2\psi}{\partial t^2} = \frac{\partial}{\partial x'}(1-x'^2)\frac{\partial\psi}{\partial x'}.$$

Putting $c = \dfrac{1}{\sqrt{\mu K}}$ so that $c$ is the velocity of propagation and taking $\psi$ to vary as $e^{i\kappa ct}$, we get

$$\frac{\partial}{\partial x'}(1-x'^2)\frac{\partial\psi}{\partial x'} + \lambda^2(x^2-x'^2)\psi = 0,$$

where $\lambda = h\kappa$.

This is exactly the same equation as we had to deal with in considering the vibrations of spheroidal layers of air. The solution is of course the same as before: $p = \lambda^2 x^2 = \kappa^2 a^2$ being determined in the case of a *complete* spheroid by the condition of convergence at the poles. The only difference is that $c$, the velocity of propagation, is different for the two problems. The *wave lengths* are the same in both cases; and have been determined in some of the more important cases on p. 92 *et seq.*

If the surface is not complete, but is bounded by a parallel of latitude $x'$, we must consider the condition at the edge. The layer of dielectric being supposed very thin, it is clear that the current ($i$) at the edge will be almost entirely tangential (along the parallel of latitude). Thus the boundary condition may be taken to be the vanishing of the component of the *magnetic* force along the parallel of latitude or $\dfrac{\partial R}{\partial x'} = 0$ at the edge.

Since $R=\sqrt{x^2-x'^2}\,.\,\psi$, the boundary condition becomes

$$(x^2-x'^2)\frac{\partial\psi}{\partial x'}-x'\psi=0.$$

At *the equator* $x'=0$, and the boundary equation is then $\frac{\partial\psi}{\partial x'}=0$, so that the conditions are exactly the same as in the sound problem with a *closed edge*. See p. 96.

*Vibrations of air contained in a spheroidal envelope.*

If $\psi$ is the velocity potential our characteristic equation is $\frac{\partial^2\psi}{\partial t^2}=c^2V^2\psi$, to solve which we take $\psi$ to vary as $e^{i\kappa ct}$ and proceed as on p. 79.

If we are dealing with a *complete* spheroidal surface we must confine ourselves to the $f$ functions, and take

$$\psi=\Sigma A_n f_n(x) f_n(x')\, e^{i\kappa ct}.$$

At a rigid boundary $\frac{\partial\psi}{\partial x}=0$, so that the boundary condition is $f_n'(x_0)=0$, which determines the admissible values of $\kappa$.

We have already shown p. 87 how the roots of this equation can be obtained in any case. The gravest note will be the fundamental one of the type $y=f_1(x)f_1(x')\,e^{i\kappa ct}$, the values of $\kappa$ being determined by roots of $f_1'(x_0)=0$.

On p. 88 we have given the lowest root of this equation for different values of the eccentricity of the bounding spheroid. We thus get these results:—

| $e$ | $\kappa a$ | $a\times$ frequency | wave length/$a$ | Ratio of frequency to case $e=0$ |
|---|---|---|---|---|
| 0 | 2·0815 | 10950 | 3·0186 | 1 |
| 0·1 | 2·0825 | 10955 | 3·0172 | 1·0005 |
| 0·2 | 2·0848 | 10960 | 3·0158 | 1·0010 |
| 0·3 | 2·0865 | 10975 | 3·0117 | 1·0024 |
| 0·4 | 2·0902 | 10993 | 3·0061 | 1·0042 |
| 0·5 | 2·0961 | 11026 | 2·9978 | 1·0070 |
| 0·6 | 2·1035 | 11064 | 2·9876 | 1·0104 |
| 0·7 | 2·1124 | 11113 | 2·9745 | 1·0150 |
| 0·8 | 2·1233 | 11170 | 2·9594 | 1·020 |
| 0·9 | 2·1364 | 11240 | 2·9411 | 1.0264 |

The most striking thing about these results is the exceedingly small influence of moderate eccentricity on the frequency.

The interval between the fundamental notes for a sphere and a spheroid of eccentricity $\frac{6}{10}$ is less than a comma.

For the sphere to sound the middle c of a piano ($c'$) its radius must be

$$a = 41{\cdot}487 \text{ centimetres} = 16\tfrac{1}{3} \text{ inches.}$$

If the eccentricity is $\frac{1}{2}$ then $a = 41{\cdot}764$ centimetres $= 16{\cdot}444$ inches.

" " $\frac{9}{10}$ " $a = 42{\cdot}57$ " $= 16{\cdot}761$ "

If we wish to find *the motion of the enclosed air due to a given normal motion of the bounding spheroid,* we have

$$\frac{\partial\psi}{\partial x} = \alpha \,.\, f(x'\kappa)\, e^{i\kappa ct} \text{ (say) when } x = x_0;$$

and so we assume $\psi = Af(x')f(x)\,e^{i\kappa ct}$, the boundary condition being $Af'(x_0) = \alpha$, $\therefore A = \dfrac{\alpha}{f'(x_0)}$ and our solution becomes $\psi = \alpha\dfrac{f(x)}{f'(x_0)}f(x')\,e^{i\kappa ct}$.

To determine the motion *of air contained between two confocal spheroids,* we must use two *independent* solutions of our fundamental equation, neither of which become infinite anywhere between the spheroids.

We may take $$\psi = [Af(x) + B\phi(x)]f(x')\,e^{i\kappa ct}.$$

When $x = x_1$ and $x = x_2$, we must have $\dfrac{\partial\psi}{\partial x} = 0$. Hence

$$Af'(x_1) + B\phi'(x_1) = 0, \quad Af'(x_2) + B\phi'(x_2) = 0.$$

Eliminating $A : B$ we get

$$\begin{vmatrix} f'(x_1), & \phi'(x_1) \\ f'(x_2), & \phi'(x_2) \end{vmatrix} = 0,$$

which is the frequency equation to determine $\kappa$.

*On the communication of vibrations from a spheroid to the surrounding gas.*

We shall suppose the disturbance due to a periodic normal motion of the surface $x = x_0$. If the normal displacement is $\phi(x')\,e^{i\kappa ct}$ we can expand $\phi(x')$ in a series of $f(x')$ functions, so that at the surface $x = x_0$ we have to deal with a boundary condition of the form $\dfrac{\partial\psi}{\partial x} = f(\lambda x')\,e^{i\kappa ct}$.

To represent the motion in the surrounding gas we want a solution appropriate to *divergent* waves; and so take $\psi = A\phi^-(z)f(z')\,e^{i\kappa ct}$, where $z = \lambda x$, $z' = \lambda x'$.

The boundary condition gives $1 = \lambda A\phi'^-(z_0)$ so that

$$\psi = \frac{\phi^-(z)}{\lambda\phi'^-(z_0)}f(z')\,e^{i\kappa ct}.$$

At a great distance from the spheroid $z$ is very large, so that $\phi^-(z)=b_0\dfrac{e^{-iz}}{z}$ approximately.

Hence at a great distance from the spheroid we may take

$$\psi=\frac{b_0}{\lambda\phi'^-(z_0)}f(z')\frac{e^{i(\kappa ct-z)}}{z}$$
$$=\frac{b_0}{\lambda\phi'^-(z_0)}f(z')\frac{e^{i\kappa(ct-a)}}{\kappa a},$$

where $a$ is the semi-major axis of the confocal ellipse through the point in question.

In estimating the energy emitted by the vibrating spheroid we may note that since there can be *on the whole* no accumulation of energy between two spheroidal surfaces, the energy transmitted across any of the confocal surfaces must be independent of $z$. Thus we may if we choose take the particular case when $z$ is very great for estimating the transmission of energy across any one of the confocal spheroidal surfaces.

Now we have just seen that when $z$ is very large we may take $\psi=Af(z')\dfrac{e^{i(\kappa ct-z)}}{z}$ where $A=b_0/\lambda\phi^{-1}(z_0)$.

The energy transmitted across a spheroid $z$ up to time $t$ is

$$W=\int dt\iint-\rho\dot{\psi}\frac{\partial\psi}{\partial n}dS.$$

With the usual notation $pdp=ada=h^2xdx$; $p'dp'=h^2x'dx'$,

$$dS=2\pi y\,.\,dp'=2\pi\frac{bb'}{h}dp'=2\pi h\sqrt{(x^2-1)(1-x'^2)}\,.\,dp'.$$

But $$p=hx\sqrt{\frac{x^2-1}{x^2-x'^2}};\quad p'=hx'\sqrt{\frac{1-x'^2}{x^2-x'^2}};\quad\therefore\ \frac{p}{p'}=\frac{x}{x'}\sqrt{\frac{x^2-1}{1-x'^2}},$$

$$\frac{\partial\psi}{\partial n}dS=\frac{\partial\psi}{\partial p}dS=\frac{p}{h^2x}\frac{\partial\psi}{\partial x}2\pi h\sqrt{(x^2-1)(1-x'^2)}\,.\,\frac{h^2x'dx'}{p'}$$
$$=2\pi h(x^2-1)\frac{\partial\psi}{\partial x}dx'.$$

$$\therefore\ W=\int dt\int_{-1}^{1}-\rho\dot{\psi}2\pi h(x^2-1)\frac{\partial\psi}{\partial x}dx'.$$

As already explained, we may take the expressions for $\dot{\psi}$ and $\dfrac{\partial\psi}{\partial x}$ corresponding to *large* values of $x$.

This gives $$\dot{\psi}=i\kappa cAf(z')\frac{e^{i(\kappa ct-z)}}{z},$$

$$\frac{\partial\psi}{\partial x}=\lambda\frac{\partial\psi}{\partial z}=-i\lambda\,.\,Af(z')\frac{e^{i(\kappa ct-z)}}{z},$$

when $z$ is large.

Hence $$W=\int dt\int_{-1}^{1}\rho\kappa c\lambda A^2\overline{f(z')}|^2\frac{e^{2i(\kappa ct-z)}}{z^2}\,.\,2\pi h\left(\frac{z^2}{\lambda^2}-1\right)dx'.$$

If we integrate $\cos m\kappa(ct-a)$ {$m$ an integer} with respect to $t$ over any number of periods we get zero: while the integral of $\cos^2\kappa(ct-a)$ or $\sin^2\kappa(ct-a)$ is $\frac{1}{2}t$, where $t$ is the time. Thus integrating over a long range of time—or rather over a range including a number of periods—we have

$$W=\tfrac{1}{2}\,.\,t\,.\,\rho\kappa c\lambda\,|A|^2\,2\pi h\left(\frac{1}{\lambda^2}-\frac{1}{z^2}\right)\int_{-1}^{1}\{f(\lambda x')\}^2\,dx',$$

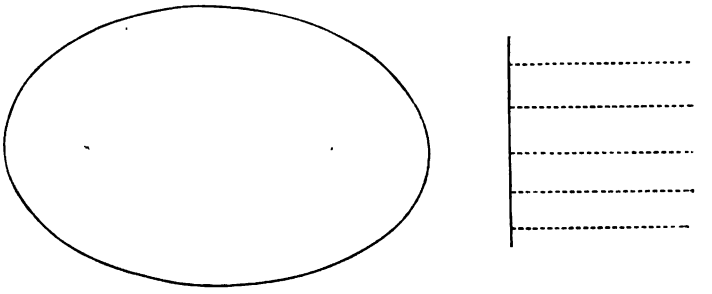
where $|A|$ is the modulus of $A=\text{mod.}\left|\dfrac{b_0}{\lambda\phi^{-1}(z_0)}\right|$.

Also since we are supposing $z$ very large $\dfrac{1}{z^2}$ may be neglected and we get (remembering that $\lambda=h\kappa$)

$$W=\pi\rho c\,|A|^2\int_{-1}^{1}\overline{f(\lambda x')}|^2\,dx'.$$

*Scattering of waves by an obstructing spheroid.*

Suppose a series of plane waves with their fronts perpendicular to the axis of a



spheroid move towards the spheroid and are scattered by it. The velocity potential of the impinging wave is

$$\phi=e^{i\kappa(ct+x_1)}=e^{i\kappa ct}\,e^{i\lambda xx'},\ \text{see p. 89},$$
$$=e^{i\kappa ct}\,\Sigma A_n f_n(\lambda x)f_n(\lambda x'),$$

where the coefficients $A_n$ are determined as on p. 90.

For the *scattered* wave we take $\psi=e^{i\kappa ct}\,\Sigma B_n\phi_n^-(\lambda x)f_n(\lambda x')$.

At the surface $x=x_0$ we must have

$$\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial x}=0,$$

$$\therefore\ A_n f_n'(\lambda x_0)+B_n\phi_n^{-1}(\lambda x_0)=0,$$

which determines $B_n$, so that the velocity potential of the scattered wave is known.

At a distance from the spheroid we may take $\phi_n^-(\lambda x)=b_0\dfrac{e^{-i\lambda x}}{\lambda x}$ giving

$$\psi=\frac{b_0e^{i(\kappa ct-\lambda x)}}{\lambda x}\,\Sigma B_n f_n(\lambda x').$$

We shall now turn to a brief discussion of the *electrical oscillations* over the surface of a *spheroid.*

Consider the case where the currents are in meridian planes through the axis of the spheroid and the magnetic force in parallels of latitude. If $a$, $b$, $c$ denote the components of the magnetic induction parallel to rectangular fixed axes, $c$ being along the axis of the spheroid, we have $c=0$, and

$$a=\sin\phi\,.\,\chi(xx'),$$
$$b=-\cos\phi\,.\,\chi(xx').$$

If as usual we take all variables proportional to $e^{ipt}$ we have $(V^2+\kappa^2)\,a=0$ in a dielectric, where $\kappa=p/V$, $V$ being the velocity of propagation in the dielectric; and $(V^2+\kappa'^2)\,a=0$ in a conductor, where $\kappa'^2=-4\pi\mu ip/\sigma$, the conductor being of specific resistance $\sigma$ and magnetic permeability $\mu$.

Since in the present case $a=\sin\phi\,.\,\chi(xx')$, we see that $\chi$ is a solution of the equation on p. 80 for the case $n=1$.

Thus *inside* the spheroid we have

$$a=A\sin\phi\,.\,f(\lambda'x)f(\lambda'x')\,e^{ipt},$$
$$b=-A\cos\phi\,.\,f(\lambda'x)f(\lambda'x')\,e^{ipt},$$
$$c=0.$$

And *outside* (in the dielectric)

$$a=B\sin\phi\,.\,\Phi^-(\lambda x)f(\lambda x')\,e^{ipt},$$
$$b=-B\cos\phi\,.\,\Phi^-(\lambda x)f(\lambda x')\,e^{ipt},$$
$$c=0.$$

Since the tangential magnetic force is continuous we have

$$\frac{A}{\mu}f(\lambda'x_0)=B\Phi^-(\lambda x_0),$$

$x=x_0$ being the bounding spheroid.

Another surface condition is got from the consideration that the electromotive intensity parallel to the spheroidal surface is continuous. Now the oscillations of the surface distribution of electricity are very rapid for ordinarily sized spheroids—the wave length being comparable with a diameter of the spheroid, and the velocity of propagation very great.

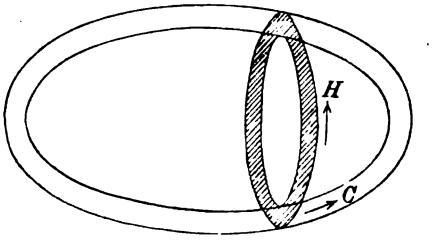
But we know that in the case of very rapid oscillations the disturbance is confined to a very thin 'skin'.

*Inside* this skin there is no E.M.F. [or there would be an electrical disturbance, which does *not* occur]. Hence for continuity of E.M.F. we see that the tangential E.M.F. vanishes at the *surface of the dielectric.*

To get an expression for the E.M.F. we make use of the circuital relation of Ampère, that the line integral of the magnetic force round a circuit $=4\pi\times$ current through the circuit.

Apply this to the ring-shaped circuit in the figure and we find, if $H$ is the resultant magnetic force and $C$ the intensity of the current,

$$4\pi C \,.\, 2\pi y \,.\, dp = \frac{d}{dx}(2\pi y \,.\, H)\, dx.$$




Now the E.M.F. parallel to $C$ in the dielectric is $\frac{4\pi}{ipK}C$, and we have seen that this vanishes, so that our surface condition is

$$\frac{d}{dx}(yH) = 0,$$

when $x = x_0$, and $$y = h\sqrt{(x^2-1)(1-x'^2)},$$

$$\therefore \frac{d}{dx_0}\left[\sqrt{x_0^2-1}\,\Phi^-(\lambda x_0)\right] = 0.$$

Now from p. 88 we see that

$$\phi^-(\lambda x) = \frac{\sqrt{x^2-1}}{\lambda^2 x^2}\left[b_0 - \frac{ib_1}{\lambda x} - \frac{b_2}{\lambda^2 x^2}\cdots\right]e^{-i\lambda x},$$

$$\therefore \sqrt{x^2-1}\,\phi^-(\lambda x) = \frac{x^2-1}{\lambda^2 x^2}\left[b_0 + \frac{ib_1}{\lambda x} - \ldots\right]e^{-i\lambda x},$$

so putting $z = \lambda x_0$, the frequency equation becomes

$$\frac{d}{dz}\cdot\left[\left(1-\frac{\lambda^2}{z^2}\right)\left(b_0 + \frac{ib_1}{z} - \ldots\right)e^{-iz}\right] = 0.$$

In the case of the sphere, $\lambda = 0$ and the series for $\phi_n^-(\lambda x)$ are finite, so that the problem is very easy. The most important case corresponds to $n = 0$, giving $p_0 = 2$, $b_1 = -b_0$, $b_2$ and all other $b$'s vanish, so that $\frac{d}{dz}\left(1-\frac{i}{z}\right)e^{-iz} = 0$, $z^2 - iz - 1 = 0$, and

$$z = \lambda x = \kappa a = \frac{i+\sqrt{3}}{2}.$$

Thus $e^{ipt} = e^{i\kappa Vt} = e^{-\frac{Vt}{2a}}e^{i\frac{\sqrt{3}V}{2a}t}$. Thus the frequency is $\frac{V\sqrt{3}}{4\pi a}$ and the modulus of decay is $\frac{2a}{V}$, so that for ordinary sized spheres the vibrations are almost "dead beat."

For values of $\lambda$ other than zero the question is not so easily answered. The modulus of $z$ for the case of the sphere just considered is unity, and in the general case the series $b_0 + \frac{ib_1}{z} + \ldots$ converges only for large values of the argument. For small

values of the eccentricity we can get $z$ from this series sufficiently accurately, as the coefficients $b_2$, $b_3$... will be very small; but for larger values of $e$ it will be necessary to replace this descending series by the *ascending* ones $f$ and $F$, in the manner indicated on p. 87.

If we wish to consider the oscillations between two confocal spheroids (conductors) we must take, in the dielectric between the conductors,

$$a=\sin\phi\,.\,f(\lambda x')\,e^{ipt}\,[C\chi^+(\lambda x)+D\chi^-(\lambda x)],$$
$$b=-\cos\phi\ldots;\ c=0.$$

By the same argument as before we must have $\frac{d}{dx}(yH)=0$ when $x=x_1$ and $x=x_2$ (the bounding surfaces),

$$\therefore\ \frac{d}{dx_1}\{[C\chi^+(\lambda x_1)+D\chi^-(\lambda x_1)]\sqrt{x_1^2-1}\}=0;\quad \frac{d}{dx_2}\{[C\chi^+(\lambda x_2)+D\chi^-(\lambda x_2)]\sqrt{x_2^2-1}\}=0,$$

and eliminating $C:D$ from these two equations we get an equation to determine the frequency and modulus of decay. The case of a thin homœoid has already been discussed on p. 96.

*On the decay of electric currents in conducting spheroids.*

First take the case when the lines of magnetic force are parallels of latitude. Then just as on p. 102 the tangential magnetic force in the conductor is $Af(\lambda' x)f(\lambda' x')\,e^{ipt}$, where $ip=-\frac{\sigma\kappa'^2}{4\pi\mu}$. This must be continuous at the surface of the conductor, and if we neglect displacement currents it is zero in the dielectric. Thus the admissible values of $\kappa$ are given by $f(\lambda' x_0)=0$, where $x=x_0$ is the surface of the conductor.

We have already shown how the roots of this equation are to be found. The tangential magnetic force is $\Sigma A_s f(\lambda_s' x)f(\lambda_s' x')\,e^{-\frac{\sigma\kappa_s'^2}{4\pi\mu}t}$, where the summation extends to the different values of $\kappa_s$ given by the roots of $f_n(\lambda' x_0)=0$. The constants $A_s$ are determined in terms of their initial values by the usual process (see p. 88).

Next take the case when the currents are in parallels of latitude. If $P$, $Q$, $R$ denote the components of E.M.F. (corresponding to $a$, $b$, $c$ of last problem) we have:—

*Inside* the spheroid
$$P=A\sin\phi\,f(\lambda' x)f(\lambda' x')\,e^{ipt},$$
$$Q=-A\cos\phi\ldots;\ R=0.$$

*Outside* the spheroid (in the dielectric)
$$P=B\sin\phi\,\Phi^-(\lambda x)f(\lambda x')\,e^{ipt},$$
$$Q=-B\cos\phi\ldots;\ R=0.$$

The continuity of the E.M.F. parallel to the spheroidal surface gives
$$Af(\lambda' x_0)=B\Phi^-(\lambda x_0).$$

Another "boundary equation" is got by considering the fact that the magnetic induction tangential to a meridian is also continuous. By making use of the circuital relation of Faraday and proceeding just as on p. 103 we get in this way

$$\frac{A}{\mu}\frac{d}{dx_0}[\sqrt{x_0^2-1}\,f(\lambda' x_0)] = B\frac{d}{dx_0}[\sqrt{x_0^2-1}\,\Phi^-(\lambda x_0)].$$

Eliminating the ratio $A:B$ from the two boundary equations we get

$$\frac{\frac{d}{dx_0}[\sqrt{x_0^2-1}\,f(\lambda' x_0)]}{\mu f(\lambda' x_0)} = \frac{\frac{d}{dx_0}[\sqrt{x_0^2-1}\,\Phi^-(\lambda x_0)]}{\Phi^-(\lambda x_0)}.$$

This equation can be considerably simplified by noting that the currents will decay so slowly that $p/V$ must be a very small quantity. Thus $\lambda$ is in this case exceedingly small.

When $\lambda = 0$ the equation on p. 88 becomes

$$z^2\frac{d^2u}{dz^2} + 2(z^2+2z)\frac{du}{dz} + (4z-p+2)\,u = 0,$$

while $p = n(n+1)$.

If we solve this in *ascending* series in the form $u = a_m z^{-m} + a_{m+1}z^{-n+1} + \ldots$, we have as the indicial equation $m^2 - 3m + 2 - n(n+1) = 0$, so that $m = n+2$, and we have $y_1 = a\,.\,\frac{e^{-i\lambda x}}{(\lambda x)^{n+2}}$ approximately when $\lambda$ is very small.

Hence when $\lambda$ is very small we have

$$\Phi_n^-(\lambda x) = a\sqrt{x^2-1}\,\frac{e^{-i\lambda x}}{(\lambda x)^{n+2}},$$

and

$$\frac{\frac{d}{dx}[\sqrt{x^2-1}\,\Phi_n^-(\lambda x)]}{\Phi_n^-(\lambda x)} = -\frac{nx}{\sqrt{x^2-1}}.$$

So that the frequency equation becomes

$$(1+n\mu)\,x_0 f_n(\lambda' x_0) + \lambda'(x_0^2-1)f_n'(\lambda' x_0) = 0,$$

and when $\mu$ is very great this is practically equivalent to $f_n(\lambda' x_0) = 0$.

Various problems on the *conduction of heat* in spheroids may be discussed on the same lines as above.

For an isotropic solid of specific heat $c$, density $\rho$, and uniform conductivity $\kappa_1$, the equation of conduction is $\frac{\partial v}{\partial t} = \frac{\kappa_1}{c\rho}V^2 v$, where $v$ is the temperature.

Suppose we have a spheroid with its surface $x = x_0$ kept at uniform temperature $v_0$ and wish to ascertain how the heat diffuses into the interior. We take $v - v_0$ to vary as $e^{-\frac{\kappa_1}{c\rho}\kappa^2 t}$ and so get $(V^2+\kappa^2)(v-v_0) = 0$,

$$\therefore\ v - v_0 = \Sigma A_n f_n(\lambda x) f_n(\lambda x')\,e^{-\frac{\kappa_1}{c\rho}\kappa^2 t},$$

which represents a temperature gradually approaching $v_0$ everywhere. The surface condition gives $f_n(\lambda x_0)=0$, and so determines the admissible values of $\kappa$.

Initially (when $t=0$) we have $v=v_0+\Sigma A_n f_n(\lambda x) f_n(\lambda x')$, and to determine the constants $A_n$ we should require to expand the initial temperature in a series of this form. (See p. 88.)

For example suppose that initially $v=v_0+Af_0(\lambda x)f_0(\lambda x')$, then at time $t$ we should have $v=v_0+Af_0(\lambda x)f_0(\lambda x')e^{-\frac{\kappa_1}{c\rho}\kappa^2 t}$. The 'modulus of decay' is $\frac{c\rho}{\kappa_1\kappa^2}=\frac{c\rho a^2}{\kappa_1 . (\kappa a)^2}$. The values of $\kappa a$ are the different roots of $f_0(\kappa a)=0$. The decay is slowest for the smallest root. Calculating this by the method of p. 87, and taking $\frac{\kappa_1}{c\rho}=1{\cdot}13$ for copper and $=0{\cdot}22$ for iron, we are led to these results:—

| $e$ | $\kappa a$ | Modulus/$a^2$ in seconds for *Copper* | Modulus/$a^2$ in seconds for *Iron* | Ratio to case $e=0$ |
|---|---|---|---|---|
| 0 | 3·1416 | ·08964 | ·4615 | 1 |
| 0·1 | 3·1538 | ·08853 | ·4547 | 0·9877 |
| 0·2 | 3·1902 | ·08696 | ·4467 | 0·9701 |
| 0·3 | 3·2528 | ·08381 | ·4295 | 0·9328 |
| 0·4 | 3·3476 | ·07898 | ·4057 | 0·8810 |
| 0·5 | 3·4785 | ·07319 | ·3760 | 0·8166 |

If the surface temperature is a given harmonic function of the time, say $v_0 e^{i\sigma t}$, we take $v$ to vary as $e^{i\sigma t}$ and get $(V^2+\kappa^2)v=0$, where $\kappa^2=-i\sigma . \frac{c\rho}{\kappa_1}$.

To represent waves of heat travelling inwards from the surface we take

$$v=\Sigma A_n \phi_n^+(\lambda x) f_n(\lambda x')\, e^{i\sigma t}.$$

The constants $A_n$ are determined from the surface condition which makes

$$v_0=\Sigma A_n \phi_n^+(\lambda x_0) f_n(\lambda x').$$

Suppose the surface temperature is $v_0 f_0(\lambda x') e^{i\sigma t}$, where $v_0$ is a constant. [We know that when $\lambda=0$, $f_n(\lambda x')=P_n(x')$, so that for small values of $\lambda$, $f_0(\lambda x')$ differs very little from a constant.]

Then the temperature at any point of the spheroid is given by

$$v=v_0\frac{\phi_0^+(\lambda x)}{\phi_0^+(\lambda x_0)} f_0(\lambda x')\, e^{i\sigma t}.$$

We have $$\kappa^2 = -i\sigma\frac{c\rho}{\kappa_1} = -i\sigma m \text{ say,}$$

$$\therefore\ \kappa = \sqrt{\frac{m\sigma}{2}}(1-i); \quad z = \lambda x = \kappa a = \sqrt{\frac{m\sigma}{2}}(1-i)\,a; \quad \lambda = a_0 e\sqrt{\frac{m\sigma}{2}}(1-i).$$

Now $$\Phi_0^+(\lambda x) = \Phi^+{}_0(z) = \frac{e^{iz}}{z}\left(1 - \frac{ib_1}{z} - \frac{b_2}{z^2}\ldots\right) = \frac{e^{iz}}{z}Re^{i\psi} \text{ say,}$$

and $$\Phi_0^+(\lambda x_0) = \frac{e^{iz_0}}{z_0}R_0 e^{i\psi_0}.$$

So $$v = v_0\frac{z_0}{z}\cdot\frac{R}{R_0}e^{i[(z-z_0)+\psi-\psi_0+\sigma t]}f_0(\lambda x')$$

$$= v_0\frac{a_0 R}{aR_0}e^{-\sqrt{\frac{m\sigma}{2}}(a_0-a)}f_0(\lambda x')\,.\,e^{i\left[-\sqrt{\frac{m\sigma}{2}}(a_0-a)+\psi-\psi_0+\sigma t\right]}.$$

This represents a wave moving inwards from the surface with velocity

$$\sqrt{\frac{2\sigma}{m}} = \sqrt{\frac{2\sigma\kappa_1}{c\rho}}.$$

The phase is not the same throughout, the change at the surface $a$ from that at $a_0$ being $\psi - \psi_0$. The amplitude also diminishes as we go inwards, its ratio at $a$ to that at the surface $a_0$ being $\frac{a_0 R}{aR_0}\,.\,e^{-\sqrt{\frac{m\sigma}{2}}(a_0-a)}$.

Making use of the relations connecting the coefficients $b_0$, $b_1\ldots$ on p. 86, and using the value of $p_0$ on p. 82 we easily get,

$$Re^{i\psi} = 1 - \left(\frac{1}{3a}\sqrt{\frac{m\sigma}{2}} - \frac{1}{12a^2}\right)a_0^2e^2 - \frac{(m\sigma)^{\frac{3}{2}}}{135\sqrt{2}\,.\,a}a_0^4e^4\ldots$$

$$-i\frac{a_0^2e^2}{3a}\sqrt{\frac{m\sigma}{2}}\left[1 - \left(m\sigma + \frac{8\sqrt{2m\sigma}}{a}\right)\frac{a_0^2e^2}{45} + \frac{2m^2\sigma^2a_0^4e^4}{3^4\,.\,5\,.\,7}\ldots\right],$$

and of course $R_0e^{i\psi_0}$ is got from this by writing $a_0$ for $a$.

Thus $$\tan\psi = -\frac{\frac{a_0^2e^2}{3a}\sqrt{\frac{m\sigma}{2}}\left[1 - \left(m\sigma + \frac{8\sqrt{2m\sigma}}{a}\right)\frac{a_0^2e^2}{45} + \frac{2m^2\sigma^2a_0^4e^4}{3^4\,.\,5\,.\,7}\ldots\right]}{1 - \left(\frac{1}{3a}\sqrt{\frac{m\sigma}{2}} - \frac{1}{12a^2}\right)a_0^2e^2 - \frac{(m\sigma)^{\frac{3}{2}}}{135\sqrt{2}\,.\,a}a_0^4e^4\ldots},$$

and $$R^2 = \left[1 - \left(\frac{1}{3a}\sqrt{\frac{m\sigma}{2}} - \frac{1}{12a^2}\right)a_0^2e^2 - \text{etc.}\right]^2 + \frac{a_0^4e^4}{9a^2}\frac{m\sigma}{2}[1 - (\quad)\ldots]^2.$$

The quantity $\frac{1}{m}$ is the "thermal diffusivity" of the substance. For copper we may take $\frac{1}{m} = 1{\cdot}13$ and for iron $\frac{1}{m} = 0{\cdot}22$. We have seen that the velocity with which the

wave travels inwards along the axis is $\sqrt{\frac{2\sigma}{m}}$. If the period were ten minutes, this would give a velocity of ·02367 centimetres per second for copper and ·004609 for iron.

The following table gives the change of phase and amplitude on the spheroid $a = 20$, the surface $a_0 = 30$ being exposed to a fluctuating temperature, the period 10 minutes.

| Eccentricity | $R/R_0$ | Ratio of amplitude at $a = 20$ to that at $a_0 = 30$ | Difference of phase $(\psi_0 - \psi)$ between $a = 20$ and $a_0 = 30$ |
|---|---|---|---|
| 0 | 1 | 0·91074 | 0 |
| 0·1 | 0·9999 | 0·91064 | 4′ |
| 0·2 | 0·99954 | 0·91032 | 14′ |
| 0·3 | 0·99826 | 0·90915 | 28′ |
| 0·4 | 0·99761 | 0·90856 | 42′ |
| 0·5 | 0·99529 | 0·90644 | 50′ |

# TRANSACTIONS

OF THE

# CAMBRIDGE

# PHILOSOPHICAL SOCIETY.

---

VOLUME XVII. PART II.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.

M.DCCC.XCIX.

## ADVERTISEMENT.

---

*The Society as a body is not to be considered responsible for any facts and opinions advanced in the several Papers, which must rest entirely on the credit of their respective Authors.*

---

The Society takes this opportunity of expressing its grateful acknowledgments to the Syndics of the University Press for their liberality in taking upon themselves the expense of printing this Part of the Transactions.

# CONTENTS.

PAGE

IV. *Certain Systems of Quadratic Complex Numbers.* By A. E. WESTERN, B.A., Trinity College, Cambridge ..... 109

V. *Partitions of Numbers whose Graphs possess Symmetry.* By Major P. A. MACMAHON, R.A., D.Sc., F.R.S., Hon. Mem. C.P.S. ..... 149

# IV. *Certain Systems of Quadratic Complex Numbers.*

By A. E. WESTERN, B.A., Trinity College, Cambridge.

[*Received* 24 October, *Read* 31 October 1898.]

1. THE object of this paper is to discuss the Theory of quadratic complex numbers from the point of view indicated by Prof. Klein in his *Lectures on Mathematics* (1894), Lecture VIII. on "Ideal Numbers."

A quantity $\phi$ which is a root of an equation

$$x^n + a_1 x^{n-1} + \dots + a_n = 0,$$

the coefficients $a_1, a_2, \dots a_n$ being rational numbers, is called an "algebraic number." (See Weber's *Lehrbuch der Algebra*, Vol. II., Chapters 16 and foll.) In particular, if the degree of the equation of lowest degree satisfied by $\phi$ is 2, $\phi$ is "an algebraic number of the second order," or more briefly, a "quadratic number." I do not propose to discuss non-integral quadratic numbers, and I shall therefore speak of "quadratic numbers," meaning thereby "quadratic integers." [$\phi$ is an "integral algebraic number" when $a_1, a_2, \dots a_n$ are all integers.]

Every quadratic number is then a root of an equation

$$x^2 + a_1 x + a_2 = 0,$$

where $a_1$ and $a_2$ are integers. Solving this equation, we obtain

$$\phi = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}.$$

Let $a_1^2 - 4a_2 = e^2 d$, where $d$ does not contain a square; then

$$\phi = \frac{-a_1 + e\sqrt{d}}{2}.$$

Note that $d \equiv 1$, 2, or 3 (mod. 4), for if $d$ were $\equiv 0$ (mod. 4) it would contain a square, contrary to hypothesis.

There are now two cases to be considered:—

(1) $a_1$ even: then $e$ is even, and $d \equiv 1$ or 2 or 3 (mod. 4), and therefore $\phi$ is of the form $x + y\sqrt{d}$, where $x$ and $y$ are integers.

(2) $a_1$ odd: then $e$ is odd, and $d \equiv 1$ (mod. 4), and therefore $\phi$ is of the form

$$\frac{x + y\sqrt{d}}{2},$$

where $x$ and $y$ are odd integers; this may also be written

$$\phi = x' + y'\frac{1+\sqrt{d}}{2},$$

$x'$ and $y'$ being integers, and $y'$ being odd. (In the volume above cited, pp. 601 and foll., Weber discusses these systems of quadratic numbers and obtains the above results.)

I shall only consider the case $d$ negative; the resulting quadratic numbers are "complex," in the usual sense of the term. And I shall call numbers of the forms $x + y\sqrt{d}$ and $x + y\frac{1+\sqrt{d}}{2}$ numbers of the first and second "type" respectively. It will be convenient to use the symbol $\theta$ in lieu of $\sqrt{d}$, or of $\frac{1+\sqrt{d}}{2}$, so that for any assigned value of $d$, and whichever type is being considered, the general form of the numbers of the system is

$$x + y\theta.$$

In the first type, $$\theta^2 - d = 0;$$

in the second, $$\theta^2 - \theta + \frac{1-d}{4} = 0.$$

By a "system," I mean the totality of numbers of the form $x + y\theta$ of a given type, for a given value of $d$.

2. The product of any two numbers of a system is a third number of the system; in fact, for a system of the first type,

$$(x + y\theta)(x' + y'\theta) = (xx' + dyy') + (xy' + x'y)\,\theta,$$

and for one of the second type,

$$(x + y\theta)(x' + y'\theta) = \left(xx' - \frac{1-d}{4}yy'\right) + (xy' + x'y + yy')\,\theta.$$

Each system is therefore, for multiplication, complete in itself.

When however the question of factorisation is considered, a system is not necessarily complete in itself. E.g.

$$6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

Thus in the system of the first type, given by $d = -5$, the number 6 can be decomposed in two distinct ways into prime factors. In face of this difficulty, Dedekind

invented his theory of "ideals" (Supplement XI. to Dirichlet's *Zahlentheorie*); and Weber, in his *Algebra*, follows the same method. This theory, owing partly to its generality and partly to the novelty of the conceptions introduced, is difficult; Klein's treatment introduces clearness and simplicity. Gauss' Composition Theory in the Theory of Binary Quadratic Forms is in fact the key to the factorisation of quadratic numbers. The connection between the Theory of Binary Quadratic Forms and the Theory of Quadratic Numbers is due to the fact that every principal form of a given determinant is the product of two conjugate quadratic numbers.

When $D \equiv 0 \pmod{4} = -4d$, the principal form of determinant $D$* is $x^2 - \frac{D}{4}y^2$, which is equal to $(x + y\sqrt{-d})(x - y\sqrt{-d})$.

And when $D \equiv 1 \pmod{4} = 1 - 4\delta$, the principal form of determinant $D$ is

$$x^2 + xy + \delta y^2,$$

which is equal to
$$\left(x + y\frac{1+\sqrt{D}}{2}\right)\left(x + y\frac{1-\sqrt{D}}{2}\right).$$

Since $D$ is to be taken negative in this paper, $d$ and $\delta$ will henceforth denote positive integers.

The product of a quadratic number and its conjugate will be called the "norm" of that quadratic number. The norm is evidently a real positive integer.

The following notation will be convenient:—

$a, b, c, \ldots$ denote quadratic numbers, $a', b', c', \ldots$ their respective conjugates, and $A, B, C, \ldots$ their respective norms, and

$$A = aa' = N(a) = N(a').$$

If then
$$a = bc,$$
it follows that
$$A = BC,$$

so that to every multiplication of quadratic numbers, corresponds a composition of quadratic forms.

Now Gauss' law of Composition asserts, that if $f$ and $f'$ are two quadratic forms of the same determinant, then the product of any two integers representable by $f$ and $f'$ respectively is representable by a definite form $F$ (which according to circumstances may belong to the same class as either $f$ or $f'$, or may belong to a different class to either). If $A$ is an integer representable by the principal form, and if $B$ and $C$ are factors of $A$ representable by forms belonging to classes other than the principal class, and if $A = BC$ it is evident that there corresponds the factorisations $a = bc$ and $a' = b'c'$, where $b$ and $b'$ are the linear factors of $B$ regarded as a quadratic form, and similarly $c$ and $c'$ of $C$.

* As suggested by Klein, I write a quadratic form $ax^2 + bxy + cy^2$: its determinant is $D = b^2 - 4ac$.

To complete a system of quadratic numbers, it is therefore necessary to introduce as "numbers" of the system the linear factors of the representative forms of each class for the given determinant: the numbers thus introduced will be called "secondary," in contradistinction to the "principal" numbers originally defined: they will be denoted by $x\lambda + y\mu$, where $x$ and $y$ are any real integers.

3. It is clear from what precedes that the properties of a system of quadratic numbers are very intimately connected with the number of classes of Quadratic Forms of the corresponding determinant $D$. Where there is only one such class, the corresponding system of quadratic numbers of the form $x + y\theta$ is complete, not only for multiplication but also for factorisation. The negative values of $D$ for which this is the case are

$$-3,\ -4,\ -7,\ -8,\ -11,\ -12,\ -16,\ -19,\ -27,\ -28,\ -43,\ -67,\ -163$$

(see table in Gauss' *Werke*, Bd. II. p. 450).

The first two of these cases are well known; $D = -3$ gives the numbers $x + y\rho$, where $\rho$ is a cube root of unity, and $D = -4$ gives the numbers $x + y\iota$, $\iota$ being a fourth root of unity.

As my object is to consider the character of the secondary numbers, in the form in which Klein has presented them, I shall set aside the systems of quadratic numbers for the values of $D$ above given, and shall devote the remainder of this paper to the consideration of the case which comes next in simplicity, when there are *two* classes only of quadratic forms.

The negative values of $D$ for which this is the case are given by $-D = 15$, 20, 24, 32, 35, 36, 40, 48, 51, 52, 60, 64, 72, 75, 88, 91, 99, 100, 112, 115, 123, 147, 148, 187, 232, 235, 267, 403, 427. (Gauss, *loc. cit.*)

When $D = -15$, the forms representing the two classes are

$$x^2 + xy + 4y^2,$$

and

$$2x^2 + xy + 2y^2.$$

The former is equal to

$$\left(x + y\frac{1+\sqrt{-15}}{2}\right)\left(x + y\frac{1-\sqrt{-15}}{2}\right);$$

the latter may be written

$$5\left(\frac{x+y}{2}\right)^2 + 3\left(\frac{x-y}{2}\right)^2,$$

and hence it is

$$\left(x\frac{\sqrt{5}+\sqrt{-3}}{2} + y\frac{\sqrt{5}-\sqrt{-3}}{2}\right)\left(x\frac{\sqrt{5}-\sqrt{-3}}{2} + y\frac{\sqrt{5}+\sqrt{-3}}{2}\right);$$

and therefore

$$\lambda = \frac{\sqrt{5}+\sqrt{-3}}{2},\quad \mu = \frac{\sqrt{5}-\sqrt{-3}}{2}.$$

The object of expressing the linear factors in this particular form is to ensure that the product of two secondary numbers should be a number of the system: this is now the case; for

$$\lambda^2 = \left(\frac{\sqrt{5}+\sqrt{-3}}{2}\right)^2 = \frac{1+\sqrt{-15}}{2} = \theta,$$

$$\lambda\mu = \left(\frac{\sqrt{5}+\sqrt{-3}}{2}\right)\left(\frac{\sqrt{5}-\sqrt{-3}}{2}\right) = 2,$$

and
$$\mu^2 = \left(\frac{\sqrt{5}-\sqrt{-3}}{2}\right)^2 = \frac{1-\sqrt{-15}}{2} = 1 - \frac{1+\sqrt{-15}}{2} = 1-\theta.$$

Also we have
$$\lambda\theta = \lambda - 2\mu, \quad \mu\theta = 2\lambda.$$

If, however, we took

$$2x^2 + xy + 2y^2 = \left(\sqrt{2}x + \frac{1+\sqrt{-15}}{2\sqrt{2}}y\right)\left(\sqrt{2}x + \frac{1-\sqrt{-15}}{2\sqrt{2}}y\right),$$

we should find that neither of these factors behaves as an integral number; for

$$\left(\frac{1+\sqrt{-15}}{2\sqrt{2}}\right)^2 = \frac{-7+\sqrt{-15}}{4} = \tfrac{1}{2}(-4+\theta),$$

which is a *non-integral* quadratic number.

In the Lecture VIII. before referred to, Klein discusses generally this question of the proper factorisation to obtain the secondary numbers, and states that "it is always possible to bring about the important result that the product of any two complex numbers" of the system of the principal and secondary numbers "will again be a complex number of the system, so that the totality of these complex numbers forms, likewise, for multiplication a complete system."

When $D = -20$, the forms representing the two classes are

$$x^2 + 5y^2,$$

and
$$2x^2 + 2xy + 3y^2.$$

The principal numbers are given by $\quad x + y\sqrt{-5},$

the secondary numbers by $\quad x\sqrt{2} + y\dfrac{1+\sqrt{-5}}{\sqrt{2}}.$

Therefore in this case

$$\lambda^2 = 2, \quad \lambda\mu = 1+\theta, \quad \mu^2 = -2+\theta, \quad \lambda\theta = -\lambda + 2\mu, \quad \mu\theta = -3\lambda + \mu;$$

the number conjugate to $\mu$ is $\quad \dfrac{1-\sqrt{-5}}{\sqrt{2}},$

$$\text{i.e. } \lambda - \mu, \text{ and } N(\mu) = \mu(\lambda - \mu) = 3.$$

This furnishes a full explanation of the paradox $6 = 2 \times 3 = (1+\theta)(1-\theta)$ mentioned above (§ 2). Neither 2, 3, $1+\theta$, nor $1-\theta$ are primes, but

$$2 = \lambda^2, \qquad 3 = \mu(\lambda - \mu),$$
$$1+\theta = \lambda\mu, \quad 1-\theta = \lambda(\lambda-\mu).$$

In like manner the values of $\lambda$ and $\mu$ may be determined for the other values of $D$ given above; it is, however, unnecessary to do so here, as the general theory of any of these systems of quadratic numbers is to a large extent independent of the particular numerical value of $D$; where this is not the case, I shall confine myself to the system $D = -20$. It should be observed that in general secondary numbers are not quadratic numbers as defined in § 1: the latter are those which have in § 2 been given the name of principal quadratic numbers. In the case $D = -20$, the only exceptions to this statement are the numbers $x\lambda$, whose square is $2x^2$, and $x\lambda\theta = x(-\lambda + 2\mu)$, whose square is $10x^2$, $x$ being a real integer.

4. A number whose norm is 1 is called a "unity"; a secondary number cannot be a unity, for 1 is representable only by forms of the principal class.

If $\alpha_0 + \alpha_1\theta$ is a unity of a system of the first type, then $\alpha_0^2 + d\alpha_1^2 = 1$.

Since $d$ is positive, and $>1$, the only solutions of this are $\alpha_0 = \pm 1$, $\alpha_1 = 0$.

Similarly for systems of the second type, we get

$$\alpha_0^2 + \alpha_0\alpha_1 + \delta\alpha_1^2 = 1.$$

Since $\delta > 1$, this gives only $\alpha_0 = \pm 1$, $\alpha_1 = 0$.

Therefore the unities of the systems now being considered are simply $+1$, and $-1$, just as in the theory of real numbers.

There are three kinds of primes in any of the systems:

(i) principal primes, which will be denoted by $p$, and whose norms $P$ are prime numbers in the ordinary sense;

(ii) secondary primes, denoted by $q$, whose norms $Q$ are also ordinary prime numbers; and

(iii) real primes, denoted by $r$, which are ordinary prime numbers not representable by either form of the determinant $D$. The norm of $r$ is $r^2$.

The primes, as thus defined, are evidently indivisible into actual factors belonging to their system. It will now be proved that they are primes in the full sense, so that any number can be expressed as the product of these primes in one and only one way.

Let $m$ be a quadratic number, principal or secondary, $M$ its norm, $m$ being such that $M$ is odd, $M_1$ a prime factor (in the ordinary sense) of $M$; and let $M = M_1M_2$. Then $M$ being a norm is representable by a quadratic form of determinant $D$, corresponding to the given system of quadratic numbers. It follows from the ordinary Theory

of Numbers that all the factors of $M$ are also representable by forms of the same determinant.

Let $H$, $H_1$, and $H_2$ be forms respectively representing $M$, $M_1$, and $M_2$. Then the theory of Composition supplies the algebraical identity

$$H(X, Y) = H_1(x, y) \times H_2(x', y'),$$

where $X$ and $Y$ are lineo-linear functions of $x$, $y$, $x'$, $y'$.

Each linear factor of $H_1(x, y)$ therefore divides $H(X, Y)$, and therefore must divide one or other of the linear factors of $H(X, Y)$.

Each linear factor of $H(X, Y)$ is therefore the product of a linear factor of $H_1(x, y)$ and a linear factor of $H_2(x', y')$; each side of this equation may then, if necessary, be multiplied by the numerical factor required to bring the linear factors of $H$, $H_1$, and $H_2$ to the correct forms (see § 3); lastly, if $X$, $Y$, $x$, $y$, $x'$, $y'$ be given the values for which the forms $H$, $H_1$, and $H_2$ respectively represent the integers $M$, $M_1$, and $M_2$, the algebraic equation becomes numerical, and gives the factorisation of $m$ in the form $m_1 m_2$. If $m_2$ is not prime, it can be similarly treated.

Thus finally we obtain a unique expression for $m$ as a product of prime factors. A method is given later for actually carrying out the process of factorisation. The proof above given is not applicable to the factorisation of the determinant itself in quadratic numbers; in most cases the determinant has a unique factorisation, but when $\frac{1}{4}D$ ($D$ being even) or $D$ (if $D$ is odd) contains a square, this is not the case; e.g. $D = -36$: then $9 = 3^2 = -\theta^2$.

Nor is it applicable to the factorisation of 2, though as a matter of fact, in most cases no exception to the general law of unique factorisation arises in connection with 2. Besides the cases where $\frac{D}{4}$ is a multiple of 4 (a particular case of $\frac{D}{4}$ containing a square) there is only one exception, which arises in the case $D = -60$; the quadratic numbers are then of the forms $x + y\sqrt{-15}$, and $x\sqrt{3} + y\sqrt{-5}$, and so

$$\lambda^2 - \mu^2 = 3 + 5 = 8,$$

and therefore the number 8 can be factorised in two distinct ways, $2 \times 2 \times 2$, and $(\lambda + \mu)(\lambda - \mu)$. In these anomalous cases Klein's method breaks down; it fails to give the ultimate prime factors. Apparently these cases can only be dealt with by Dedekind's method. Putting aside those values of $D$ with which we cannot deal, the following values of $-D$ remain to be studied: 15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232, 235, 267, 403, 427.

The systems arising from the latter values of $D$ can be divided into three sets:

(i) Those in which $2 = \lambda^2$; these are given by $-D = 20$, 24, 40, 52, 88, 148, and 232.

(ii) That in which $2 = \lambda\mu$, being the case $D = -15$.

(iii) Those in which 2 is a prime; these are given by $-D=35$, 51, 91, 115, 123, 187, 235, 267, 403, 427.

The numerical values of $\theta$, $\lambda$, and $\mu$, and of their norms, and the values of $\lambda\theta$, $\mu\theta$, etc., are given for these 18 values of $D$ in Tables I. and II., at the end of the paper.

5. All the elementary theorems of the ordinary Theory of Numbers in regard to prime numbers and divisibility are true for the 18 systems to which I now confine myself, since for each of them the law of unique factorisation exists. I shall therefore freely adopt, without definition, the technical terms, such as "modulus," "residue," "congruence," of the ordinary Theory of Numbers, and the usual notation connected therewith.

The operation of multiplication can be applied to any numbers of a system, whether secondary or principal. It follows at once from the theory of composition, applied to the case of determinants with two classes, that the product of two principal numbers, or of two secondary numbers, is a principal number, and that the product of a principal and a secondary number is a secondary number. Examples of these laws are given in § 3, for the cases $D=-15$, and $-20$. On the other hand addition only operates between two numbers of the same class, either both principal or both secondary. For the quantity obtained by adding together a principal and a secondary number does not belong to and has no necessary connection with the system of principal and secondary numbers, and must therefore be considered as irrelevant to the present subject.

6. Residues.—In accordance with the principle above stated, a principal number cannot be congruent to a secondary number. Complete sets of principal residues and of secondary residues to a given modulus are required: the most convenient complete sets are given by the following formulae, which relate to the case $D=-20$; the method of obtaining them is the same for each system, but the actual results differ slightly from each other.

I. Principal modulus, $n=g(x+y\theta)$, where $x$ is prime to $y$; let $M$ be $x^2+5y^2$, the norm of $x+y\theta$.

(1) Principal residues:—A complete set is given by $s+t\theta$, where

$$s=0,\ 1,\ \ldots\ldots\ (Mg-1),$$

$$t=0,\ 1,\ \ldots\ldots\ (g-1).$$

(2) Secondary residues:—A complete set is given by $s\lambda+t\mu$, where

$$s=0,\ 1,\ \ldots\ldots\ \left(\frac{M}{k}g-1\right),$$

$$t=0,\ 1,\ \ldots\ldots\ (kg-1),$$

$k$ being 1 or 2, according as $x+y$ is odd or even, i.e. according as $M$ is odd or even.

II. Secondary modulus, $n = g(x\lambda + y\mu)$, where $x$ is prime to $y$. $M = 2x^2 + 2xy + 3y^2$, the norm of $x\lambda + y\mu$.

(1) Principal residues:—A complete set is given by $s + t\theta$, where

$$s = 0,\ 1,\ \ldots\ldots\ (Mg - 1),$$
$$t = 0,\ 1,\ \ldots\ldots\ (g - 1).$$

(2) Secondary residues:—A complete set is given by $s\lambda + t\mu$, where

$$s = 0,\ 1,\ \ldots\ldots\ \left(\frac{M}{k}g - 1\right),$$
$$t = 0,\ 1,\ \ldots\ldots\ (kg - 1),$$

$k$ being 1 or 2, according as $y$ is odd or even, i.e. according as $M$ is odd or even.

It should be noticed that in all cases, the number of residues in a complete set is $Mg^2$, which is the norm of the modulus.

The proof of I. (2) will be given as a specimen. If $s\lambda + t\mu$ is a multiple of $n$, then

$$s\lambda + t\mu = g(x + y\theta)(u\lambda + v\mu).$$

Now $\lambda\theta = -\lambda + 2\mu$, $\mu\theta = -3\lambda + \mu$, so equating coefficients of $\mu$ on each side, we get

$$t = g\{v(x + y) + 2uy\};$$

so, $k$ being 1 or 2, according as $x + y$ is odd or even,

$$t \equiv 0 \ (\text{mod. } kg) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Also,
$$(s\lambda + t\mu)(x - y\theta) = gM(u\lambda + v\mu),$$
therefore
$$s(x + y) + t(3y) \equiv 0 \ (\text{mod. } gM),$$
$$s(-2y) + t(x - y) \equiv 0 \ (\text{mod. } gM).$$

Find $\xi$ and $\eta$, so that $(x + y)\eta - 2y\xi = k$; then we get

$$ks + t(3y\eta + x\xi - y\xi) \equiv 0 \ (\text{mod. } gM).$$

Now $M = x^2 + 5y^2$, where $x$ and $y$ are not both even, for they are coprime: if one is odd, the other even, $M$ is odd, and $k = 1$; if both $x$ and $y$ are odd, $M$ is even, and $k = 2$; in either case $M \equiv 0$ (mod. $k$).

Therefore, dividing the last congruence throughout by $k$,

$$s + \frac{t}{k}(3y\eta + x\xi - y\xi) \equiv 0 \left(\text{mod. } g\frac{M}{k}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Congruences (i) and (ii) are the necessary and sufficient conditions that $s\lambda + t\mu$ should be a multiple of $n$.

No two numbers of the set $s\lambda + t\mu$, where

$$s = 0,\ 1,\ \ldots\ldots\ \left(\frac{M}{k}g - 1\right),$$
$$t = 0,\ 1,\ \ldots\ldots\ (kg - 1),$$

are therefore congruent for the modulus $n$, and every other secondary number is congruent to one of them: in other words, this set is complete.

In the important case of the modulus being prime, the results are simpler, and admit of being stated generally for all the systems.

(i) Modulus $n$, a principal or secondary prime, whose norm is $N$.

Principal residues:—0, 1, 2, ....., $N-1$.

Secondary residues:—0, $\lambda$, $2\lambda$, ....., $(N-1)\lambda$, provided that $n$ is not $\lambda$.

(ii) Modulus $r$, a real prime.

Principal residues are given by $s+t\theta$,

Secondary residues by $s\lambda+t\mu$,

where
$$s=0,\ 1,\ 2,\ \ldots\ldots,\ r-1,$$
$$t=0,\ 1,\ 2,\ \ldots\ldots,\ r-1.$$

7. Factorisation of numbers.—Each real prime factor of a real number can either be represented by one of the quadratic forms connected with the system, in which case it is the product of two conjugate prime factors, or it is a real prime of the system.

To factorise $n=x+y\theta$, or $x\lambda+y\mu$, where $x$ and $y$ are coprime. Express $N$, the norm of $n$, as the product of real prime factors $A$, $B$, ..... Each of these numbers is representable by one of the quadratic forms connected with the system; therefore $A=aa'$, $B=bb'$, ..., where $a$, $a'$, ... are primes of the system.

By the method shewn in the previous section, calculate the residue of $n$ (mod. $a$); if this residue is 0, $a$ is a factor of $n$; if it is not 0, $a'$ must be a factor of $n$. Similarly the other factors of $n$ are determined.

Example, in the system $D=-20$.
$$55(61+2\theta).$$
$$5=-\theta^2,\ 11 \text{ is a prime.}$$

The norm of $61+2\theta$ is $61^2+5.2^2=3741=3.29.43$.
$$3=\mu(\lambda-\mu),$$
$$29=3^2+5.2^2=(3+2\theta)(3-2\theta),$$
$$43=2.4^2+2.4.1+3.1^2=(4\lambda+\mu)(5\lambda-\mu).$$

If
$$\theta\equiv x \text{ (mod. } 3+2\theta),$$
$$(\theta-x)(3-2\theta)\equiv 0 \text{ (mod. } 29);$$

hence
$$\begin{cases}3x-10\equiv 0 \text{ (mod. } 29)\\ 2x+\ \ 3\equiv 0 \text{ (mod. } 29),\end{cases}$$

and so
$$x\equiv 13 \text{ (mod. } 29).$$

Then
$$61+2\theta\equiv 61+2x\equiv 87\equiv 0 \text{ (mod. } 3+2\theta).$$

Similarly $61+2\theta$ is a multiple of $\lambda-\mu$, and of $4\lambda+\mu$.

Therefore the number $55(61+2\theta)$ is $-11\theta^2(\lambda-\mu)(4\lambda+\mu)(3+2\theta)$.

8. Congruences.—Precisely as in the ordinary Theory of Numbers, Lagrange's theorem as to congruences may be proved:—a congruence of the $n$th degree, the modulus being prime, cannot have more than $n$ incongruent roots.

And if such a congruence appears to have more than $n$ incongruent roots, it is an identical congruence.

The linear congruence

$$ex \equiv f \pmod{n},$$

$n$ being any number, and $e$ being prime to $n$, has one and only one root.

$E$ and $N$ being the respective norms of $e$ and $n$, real integers $\xi$ and $\eta$ can be found such that

$$E\eta - N\xi = 1.$$

Then since

$$E\eta \equiv 1 \pmod{n},$$

$$ee'\eta f \equiv f \pmod{n},$$

where $e'$ is the conjugate of $e$.

The solution is therefore $x \equiv e'\eta f \pmod{n}$.

9. Fermat's Theorem.—I. For powers of a principal number $h$, prime to the modulus $n$, which is a prime of any one of the three kinds.

Let $K_1, K_2, \ldots K_{N-1}$ denote a complete set (except 0) of principal residues to the modulus $n$. Then $hK_1, hK_2, \ldots hK_{N-1}$ is also a complete set.

Therefore

$$h^{N-1}(K_1K_2 \ldots K_{N-1}) \equiv K_1K_2 \ldots K_{N-1} \pmod{n},$$

and so

$$h^{N-1} \equiv 1 \pmod{n}.$$

II. For powers of a secondary number $j$, prime to the modulus $n$.

In any system, a secondary number $j_0$ can be found whose square is a real integer, and which is prime to $n$.

In systems of the first type, provided $n$ is odd, $j_0$ may be taken to be $\lambda$, and then $j_0^2 = 2$.

In systems of the second type, $j_0 = \lambda + \mu$, or $\lambda - \mu$, one or other of which is prime to $n$.

Then $j_0^2 = f$, where $f$ may be found for each system from Table I.

Then

$$jK_1, jK_2, \ldots jK_{N-1},$$

and

$$j_0K_1, j_0K_2, \ldots j_0K_{N-1},$$

are two complete sets of secondary residues to the modulus $n$.

And therefore

$$j^{N-1} \equiv j_0^{N-1} \pmod{n}.$$

For systems of the first type, we get

$$j^{N-1} \equiv 2^{\frac{N-1}{2}} \pmod{n}$$
$$\equiv (2/N) \pmod{n}$$
$$\equiv (-1)^{\frac{N^2-1}{8}} \pmod{n}.$$

For systems of the second type,

$$j^{N-1} \equiv f^{\frac{N-1}{2}} \pmod{n}$$
$$\equiv (f/N) \pmod{n}.$$

When a particular system is specified, the expression of Fermat's theorem can be simplified. Thus for the system $D=-20$, the theory of generic characters shews that $P$ (the norm of an odd principal prime $p$), being an odd integer representable by the principal form, satisfies the congruence

$$P \equiv 1 \pmod{4};$$

and that $Q$ (the norm of an odd secondary prime $q$) satisfies $Q \equiv -1 \pmod{4}$.

Applying the general theorem that has just been proved

$$j^{N-1} \equiv (-1)^{\frac{N^2-1}{8}} \pmod{n},$$

first to the case of $n=p$, we obtain

$$j^{P-1} \equiv +1 \text{ or } -1 \pmod{p},$$

according as $$P \equiv 1 \text{ or } 5 \pmod{8}.$$

And therefore $$j^{P-1} \equiv (-1)^{\frac{P-1}{4}} \pmod{p}.$$

Similarly $$j^{Q-1} \equiv (-1)^{\frac{Q+1}{4}} \pmod{q},$$

and $$j^{R-1} \equiv 1 \pmod{r},$$

since $$R = r^2 \equiv 1 \pmod{8}.$$

10. In the theory of real numbers, if $a$ is an odd number

$$a^2 \equiv 1 \pmod{8}.$$

I propose to consider the analogue of this for quadratic numbers, but here and in the sequel, whenever the properties of 2 are in question, I shall confine the discussion to the seven systems mentioned in § 4, for which $2=\lambda^2$. In these cases $\theta=\sqrt{-d}$ where $d=\frac{-D}{4}$.

Numbers will be called odd, semi-even or even, according as they are prime to $\lambda$, a multiple of $\lambda$ but not of 2, or a multiple of 2. As above $h$ denotes a principal number, $j$ a secondary number.

If $h$ is odd, $\quad h \equiv 1 \pmod{\lambda}$,

that is $\quad h = 1 + \lambda k$;

therefore $\quad h^2 = 1 + 2\lambda k + 2k^2$

$$\equiv 1 \pmod{2}.$$

To proceed further, with higher powers of $\lambda$ for moduli, it is necessary to distinguish the cases where $d$ is odd, from those where it is even.

First, the case $d$ odd; then $\theta = \sqrt{-d}$ is odd.

So, $h$ being odd $\quad \equiv 1$ or $\theta \pmod{2}$,

that is $\quad h = 1 + 2k$, or $\theta + 2k$;

therefore $\quad h^2 = 1 + 4k + 4k^2$, or $-d + 4\theta k + 4k^2$.

Now, if $k$ is odd, so are $\theta k$ and $k^2$; and if $k$ is a multiple of $\lambda$, so are $\theta k$ and $k^2$: in either case $k^2 + k$ and $k^2 + \theta k$ are multiples of $\lambda$.

And so $\quad h^2 \equiv 1$ or $-d \pmod{4\lambda}$,

according as $\quad h \equiv 1$ or $\theta \pmod{2}$.

Similarly when $d$ is even, and, as before, $h$ is odd,

$$h^2 \equiv 1 \text{ or } 1 - d + 2\theta \pmod{4\lambda},$$

according as $\quad h \equiv 1$ or $1 + \theta \pmod{2}$.

Similar results hold for squares of odd secondary numbers. First, the case $d$ odd.

Then $$\mu = \frac{1 + \sqrt{-d}}{\sqrt{2}}, \text{ so } \mu^2 = \frac{1-d}{2} + \theta.$$

As before we get $j^2 \equiv \mu^2 \pmod{4\lambda}$, if $j \equiv \mu \pmod{2}$, and $j^2 \equiv (\lambda - \mu)^2 \pmod{4\lambda}$, if $j \equiv \lambda - \mu \pmod{2}$.

And so $$j^2 \equiv \frac{1-d}{2} + \theta, \text{ or } \frac{1-d}{2} - \theta \pmod{4\lambda},$$

according as $\quad j \equiv \mu$ or $\lambda - \mu \pmod{2}$.

Secondly, in the case $d$ even $\quad \mu = \sqrt{-\frac{d}{2}}$,

and so $$j^2 \equiv -\frac{d}{2} \text{ or } 2 - \frac{d}{2} - 2\theta \pmod{4\lambda},$$

according as $\quad j \equiv \mu$ or $\lambda - \mu \pmod{2}$.

It is worthy of notice that the squares of odd numbers (principal or secondary) in any one of these quadratic systems can be congruent only to 4 out of the 16 odd

residues to the modulus $\lambda^5$, i.e. $4\lambda$; a result remarkably similar to the corresponding fact in the system of real numbers, viz. that all odd squares are congruent to 1 out of the 4 odd residues to the modulus $2^3$, i.e. 8.

It will be convenient for future use to gather up in tabular form these results and some other similar ones, for the particular system $D = -20$.

| | Residue of Number to mod. 2 | Residue of Square to mod. 4 | Residue of Square to mod. $4\lambda$ |
|---|---|---|---|
| Principal numbers | $0$ | $0$ | $0$ or $4$ |
| | $1+\theta$ | $2\theta$ | $\pm 2\theta$ |
| | $1$ | $1$ | $1$ |
| | $\theta$ | $-1$ | $3$ |
| Secondary numbers | $0$ | $0$ | $0$ or $4$ |
| | $\lambda$ | $2$ | $\pm 2$ |
| | $\mu$ | $2+\theta$ | $-2+\theta$ |
| | $\lambda+\mu$ | $-2-\theta$ | $-2-\theta$ |

11. Quadratic Congruences to a prime modulus.—Since the square of either a principal or a secondary number is a principal number, the general form of congruence to be considered is

$$x^2 \equiv h \pmod{n},$$

$h$ being a principal number prime to $n$, and $n$ an odd prime of any of the three kinds. According to circumstances, this congruence may have either principal solutions only, or secondary solutions only, or neither, or both.

(i) Principal solutions:—the necessary and sufficient condition for solubility is

$$h^{\frac{N-1}{2}} \equiv 1 \pmod{n}.$$

This condition is necessary, for

$$h^{\frac{N-1}{2}} \equiv x^{N-1} \equiv 1 \pmod{n}; \quad (\S\ 9,\ \text{I.})$$

and also sufficient, for taking the complete set of principal residues to the modulus $n$ in the form

$$\pm K_1,\ \pm K_2,\ \ldots\ldots \pm K_{\frac{N-1}{2}},$$

and squaring each, we obtain $\frac{N-1}{2}$ different residues of squares of principal numbers; each of them satisfies $h^{\frac{N-1}{2}} \equiv 1$ (mod. $n$), a congruence (in $h$ as unknown) which cannot have more than $\frac{N-1}{2}$ roots; therefore every value of $h$ such that $h^{\frac{N-1}{2}} \equiv 1$ (mod. $n$) is congruent to the squares of two of the principal residues of $n$.

Further the $\frac{N-1}{2}$ residues which are not congruent to squares of principal numbers are the roots of

$$h^{\frac{N-1}{2}} \equiv -1 \text{ (mod. } n).$$

The symbol $(h/n)$ will be used to denote the least residue (either $+1$, or $-1$) of $h^{\frac{N-1}{2}}$ (mod. $n$). This must be distinguished from the analogous symbol $(A/B)$ in the ordinary Theory of Numbers. It should be noticed that $(h/n)$ has no meaning, unless $h$ is a principal number, *and* $n$ is a prime number.

(ii) Secondary solutions of $x^2 \equiv h$ (mod. $n$).

Then
$$h^{\frac{N-1}{2}} \equiv x^{N-1} \equiv \pm 1 \text{ (mod. } n),$$
the ambiguous sign depending on the system of numbers and the value of $n$ (see § 9, II.).

As before it may be proved that the condition $h^{\frac{N-1}{2}} \equiv \pm 1$ (mod. $n$), (as the case may be) is sufficient as well as necessary for the existence of secondary roots of the congruence.

For systems of the first type, this ambiguous sign may be expressed as $(-1)^{\frac{N^2-1}{8}}$ (§ 9, II.). Accordingly if $N \equiv \pm 1$ (mod. 8), the congruence $x^2 \equiv h$ (mod. $n$) is soluble both in principal and in secondary numbers, provided that $(h/n) = +1$; but it is soluble in neither, if $(h/n) = -1$. On the other hand, if $N \equiv \pm 3$ (mod. 8) the congruence is soluble in principal numbers only if $(h/n) = +1$, and in secondary numbers only if $(h/n) = -1$.

12. The value of the symbol $(h/n)$ can always be expressed in terms of the corresponding symbol in the ordinary theory of numbers, and so, when $h$ and $n$ are given, its actual value, $+1$ or $-1$, can easily be determined. The proof of this statement differs according as $n$ is a real prime or not.

First, let $n$ be a principal or secondary prime. Then (§ 6) a real number $h_0$ may be found congruent to $h$ (mod. $n$). Hence $(h/n) \equiv h^{\frac{N-1}{2}} \equiv h_0^{\frac{N-1}{2}}$ (mod. $n$); but

$$h_0^{\frac{N-1}{2}} \equiv (h_0/N) \text{ (mod. } N),$$

and so, since $N$ is a multiple of $n$,

$$h_0^{\frac{N-1}{2}} \equiv (h_0/N) \text{ (mod. } n);$$

therefore
$$(h/n) = (h_0/n) = (h_0/N).$$

Secondly, let $n = r$, a real prime.

Let $h = x + y\theta$, and its conjugate $h' = x + y\theta'$.

If the system is of the first type, $\theta = \sqrt{-d}$, and so

$$\theta^r = (-d)^{\frac{r-1}{2}}\,\theta \equiv (-d/r)\,\theta \pmod{r}.$$

Now $(-d/r) = -1$, for $r$ is not representable by any form of determinant $-4d$; and $-\theta = \theta'$; therefore $\qquad \theta^r \equiv \theta' \pmod{r}$.

The same result holds for systems of the second type: then

$$2\theta = 1 + \sqrt{D}, \quad 2\theta' = 1 - \sqrt{D}:$$

so
$$\begin{aligned} 2^r\theta^r = (1+\sqrt{D})^r &\equiv 1 + D^{\frac{r-1}{2}}\sqrt{D} \pmod{r} \\ &\equiv 1 - \sqrt{D} \pmod{r} \\ &\equiv 2\theta' \pmod{r}; \end{aligned}$$

also,
$$2^r \equiv 2 \pmod{r},$$

and hence
$$\theta^r \equiv \theta' \pmod{r}.$$

Then, in either case, $\quad h^r = (x+y\theta)^r \equiv x^r + y^r\theta^r \pmod{r}$.

But $\quad x^r \equiv x, \; y^r \equiv y, \text{ and } \theta^r \equiv \theta' \pmod{r},$

so
$$h^r \equiv x + y\theta' \equiv h' \pmod{r};$$

therefore
$$h^{r+1} \equiv hh' \equiv H \pmod{r}.$$

And so finally, $\quad (h/r) \equiv h^{\frac{r^2-1}{2}} \equiv H^{\frac{r-1}{2}} \equiv (H/r) \pmod{r}$.

It should be observed that, if both the numbers $a$ and $b$ in the symbol $(a/b)$ are real, the symbol is still different from $(a/b)$. For in that case,

$$(a/b) \equiv a^{\frac{b^2-1}{2}} \equiv (a^{b-1})^{\frac{b+1}{2}} \equiv 1 \pmod{b},$$

while $(a/b)$ of course may be either $\pm 1$.

It is evident that $\quad (h_1h_2h_3\ldots/n) = (h_1/n)\,(h_2/n)\,(h_3/n)\ldots.$

13. Laws of Quadratic Reciprocity between two principal (including real) primes exist for all systems of principal quadratic numbers; these laws are analogous to, and indeed are deduced from, the Law of Quadratic Reciprocity in the ordinary Theory of Numbers. In determining these laws, systems of the first and second types must be separately considered, and the former will have to be subdivided according as

$$d = \frac{-D}{4} \equiv 1 \pmod{2}, \text{ or } 2 \pmod{4}.$$

For the reason appearing in § 4, systems for which $d \equiv 0 \pmod{4}$ are omitted.

The following notation will be used, in discussing the laws of reciprocity for systems of the first type: $p = x + y\theta$, and $p' = x' + y'\theta$ denote two (non-conjugate) principal primes; $P = x^2 + dy^2$ and $P' = x'^2 + dy'^2$ are their respective norms.

Then $$PP' = (x^2 + dy^2)(x'^2 + dy'^2) = X^2 + dY^2,$$
where $$X = xx' + dyy', \quad Y = x'y - xy'.$$

Now we have identically
$$x(x' + y'\theta) = xx' + dyy' + y'\theta(x + y\theta),$$
that is $$xp' \equiv X \pmod{p},$$
and so $$(x/p)(p'/p) = (X/p).$$

Since $x$ and $X$ are real numbers, $(x/p) = (x/P)$, and $(X/p) = (X/P)$.

Therefore $$(p'/p) = (x/P)(X/P).$$

Similarly $$(p/p') = (x'/P')(X/P'),$$
and so $$(p/p')(p'/p) = (x/P)(x'/P')(X/PP').$$

This formula is true for all systems of the first type: in order to evaluate the right-hand side of the last equation, we must consider separately the cases when $d$ is even or odd.

14. First, let $d$ be odd. Then since $P = x^2 + dy^2$, either $x$ is even and $y$ odd, or vice-versa. If $p \equiv 1 \pmod{2}$, then $y \equiv 0 \pmod{2}$, and $P \equiv x^2 \equiv 1 \pmod{4}$; but if $p \equiv \theta \pmod{2}$, then $x \equiv 0 \pmod{2}$, and $P \equiv d \pmod{4}$.

In order to evaluate $(x/P)$, three cases must be treated:

(i) $x$ odd, and $y$ even.

(ii) $x = 2\xi$, where $\xi$ is odd, and $y$ odd.

(iii) $x = 2^\mu\xi$, where $\xi$ is odd, and $\mu > 1$, and $y$ odd.

(i) Here $p \equiv 1 \pmod{2}$: then $(x/P) = (P/x)$, since $P \equiv 1 \pmod{4}$
$$= (x^2 + dy^2/x) = (dy^2/x) = (d/x)$$
$$= (-1)^{\frac{x-1}{2}\cdot\frac{d-1}{2}}(x/d).$$

(ii) $x = 2\xi$, where $\xi$ is odd, and $y$ odd.

Then $$P = 4\xi^2 + dy^2 \equiv 4 + d \pmod{8},$$
and so $$(x/P) = (2/P)(\xi/P) = (2/P)(-1)^{\frac{\xi-1}{2}\cdot\frac{P-1}{2}}(P/\xi).$$

Now $$(2/P) = (2/4 + d) = -(2/d),$$
and $$\frac{P-1}{2} \equiv \frac{d-1}{2} \pmod{2},$$
and $$(P/\xi) = (d/\xi).$$

Therefore
$$(x/P) = -(2/d)(-1)^{\frac{\xi-1}{2}\cdot\frac{d-1}{2}}(d/\xi)$$
$$= -(2/d)(\xi/d) = -(x/d).$$

(iii) $x = 2^\mu\xi$, where $\mu > 1$, and $\xi$ is odd: also $y$ is odd.

Then
$$P \equiv d \pmod{8},$$

therefore
$$(x/P) = (2/P)^\mu(\xi/P) = (2/P)^\mu(-1)^{\frac{\xi-1}{2}\cdot\frac{P-1}{2}}(P/\xi)$$
$$= (2/d)^\mu(-1)^{\frac{\xi-1}{2}\cdot\frac{d-1}{2}}(d/\xi)$$
$$= (2^\mu/d)(\xi/d) = (x/d).$$

Cases (ii) and (iii) can be summed up in one theorem: if $x$ is even, then
$$(x/P) = (-1)^{\frac{x}{2}}(x/d).$$

15. While continuing to treat the case of $d$ odd, we are now in a position to evaluate $(p/p')(p'/p)$. Since $PP' = X^2 + dY^2$, the results of the last paragraph give the value of $(X/PP')$, as well as of $(x/P)$ and $(x'/P')$.

Three cases must be separately considered:

(i) $p \equiv p' \equiv 1 \pmod{2}$;

(ii) $p \equiv 1 \pmod{2}$, $p' \equiv \theta \pmod{2}$;

(iii) $p \equiv p' \equiv \theta \pmod{2}$.

It will suffice to give the detailed working in the first of these cases only. In this $x$ and $x'$ are odd, $y$ and $y'$ are even. So $X = xx' + dyy' \equiv xx' \pmod{4}$, and therefore $X$ is odd, and $Y$ is even. Hence the result of § 14 (i) gives
$$(x/P) = (-1)^{\frac{x-1}{2}\cdot\frac{d-1}{2}}(x/d),$$
$$(x'/P') = (-1)^{\frac{x'-1}{2}\cdot\frac{d-1}{2}}(x'/d),$$
and
$$(X/PP') = (-1)^{\frac{X-1}{2}\cdot\frac{d-1}{2}}(X/d),$$
$$= (-1)^{\frac{xx'-1}{2}\cdot\frac{d-1}{2}}(xx'/d).$$

Therefore
$$(p/p')(p'/p) = (x/P)(x'/P')(X/PP')$$
$$= (-1)^{\frac{d-1}{2}\cdot\left(\frac{x-1+x'-1+xx'-1}{2}\right)}.$$

Now
$$x - 1 + x' - 1 + xx' - 1 = (x+1)(x'+1) - 4 \equiv 0 \pmod{4}.$$

And so the index of $-1$ on the right side of the last equation is even, and therefore, when $p \equiv p' \equiv 1 \pmod{2}$,
$$(p/p')(p'/p) = 1.$$

The result when $p \equiv 1 \pmod{2}$, $p' \equiv \theta \pmod{2}$ is similarly found to be
$$(p/p')(p'/p) = (-1)^{\frac{x-1}{2}\cdot\frac{d-1}{2}+\frac{y}{2}};$$

that is, if $d \equiv 1$ (mod. 4), $\qquad (-1)^{\frac{y}{2}}$,

but if $d \equiv 3$ (mod. 4), $\qquad (-1)^{\frac{x+y-1}{2}}$.

And when $p \equiv p' \equiv \theta$ (mod. 2), $\quad (p/p')(p'/p) = (-1)^{\frac{x+x'}{2}+\frac{d-1}{2}\cdot\frac{y+y'}{2}}$;

that is, if $d \equiv 1$ (mod. 4), $\qquad (-1)^{\frac{x+x'}{2}}$,

but if $d \equiv 3$ (mod. 4), $\qquad (-1)^{\frac{x+x'+y+y'}{2}}$.

Finally, the Laws of Quadratic Reciprocity for systems of the first type, for which $d \equiv 1$ (mod. 4), can be summed in the following simple and beautiful shape:—

If either $p$ or $p' \equiv 1$ (mod. 2), then $(p/p')(p'/p) = (-1)^{\frac{yy'}{2}}$.

If both $p$ and $p' \equiv \theta$ (mod. 2), then $(p/p')(p'/p) = (-1)^{\frac{x+x'}{2}}$.

This law holds for the systems $D = -20, -52, -148$ (for which $d = 5, 13, 37$ respectively); but it also holds for all other systems of principal quadratic numbers of the first type for which $d \equiv 1$ (mod. 4), as the existence of secondary numbers is irrelevant to its proof.

In the case $d \equiv 3$ (mod. 4) we similarly have:—

If either $p$ or $p' \equiv 1$ (mod. 2), $(p/p')(p'/p) = (-1)^{\frac{x(x-1)y'+x'(x'-1)y+yy'}{2}}$.

If both $p$ and $p' \equiv \theta$ (mod. 2), $(p/p')(p'/p) = (-1)^{\frac{x+x'+y+y'}{2}}$.

16. The case $d \equiv 2$ (mod. 4) remains to be treated, in order to complete the discussion as regards systems of the first type.

The procedure is similar to what has been given, so that there is no need to set out the working, which is somewhat lengthy, owing to the number of different cases.

If either $p$ or $p' \equiv 1$ (mod. 2), $\qquad (p/p')(p'/p) = (-1)^{\frac{(x-1)y'+(x'-1)y+yy'}{2}}$.

If both $p$ and $p' \equiv 1 + \theta$ (mod. 2), $(p/p')(p'/p) = (-1)^{\frac{y+y'}{2}}$.

17. It is a remarkable fact that the laws of reciprocity between a principal prime and a real prime, or between two real primes, are the same as would be obtained from the law between two principal primes by making one or both of them become real primes. That is to say, if in the formulæ for $(p/p')(p'/p)$ we write $x' = r$, $y' = 0$, we obtain the true expression for $(p/r)(r/p)$.

For, in every system of the first type, $(p/r) = (P/r)$ and $(r/p) = (r/P)$, (§ 12); and so

$$(p/r)(r/p) = (P/r)(r/P) = (-1)^{\frac{P-1}{2}\frac{r-1}{2}}.$$

Now if $d \equiv 1$ (mod. 4), $P \equiv 1$ (mod. 4), (§ 14), and hence

$$(p/r)(r/p) = +1.$$

This is what is obtained from

$$(p/p')(p'/p) = (-1)^{\frac{yy'}{2}}$$

when we put $r$ for $x'$, 0 for $y'$; but it need hardly be said that this substitution of $r$ for $p'$ does not prove the result.

Next, in the cases $d \equiv 2$ or 3 (mod. 4), $P \equiv 1$ or 3 (mod. 4) according as $y \equiv 0$ or 1 (mod. 2): that is

$$\frac{P-1}{2} \equiv y \text{ (mod. 2)}.$$

And so we get $$(p/r)(r/p) = (-1)^{y\frac{r-1}{2}}.$$

And this is what we get from the formulæ for $d \equiv 3$ (mod. 4) (§ 15), and for $d \equiv 2$ (mod. 4) (§ 16) on putting $r$ for $x'$, 0 for $y'$.

In the case of two real primes, $r$ and $r'$, since $(r/r') = +1$, and $(r'/r) = +1$, we get $(r/r')(r'/r) = +1$, which agrees with the formulæ for $(p/p')(p'/p)$.

18. Similar but more complicated laws hold for systems of the second type. Here, if

$$p = x + y\theta, \qquad p' = x' + y'\theta,$$

then $$P = x^2 + xy + \delta y^2, \quad P' = x'^2 + x'y' + \delta y'^2.$$

Also, $p_0'$ being the conjugate of $p'$, $p_0' = x' + y'\theta' = x' + y' - y'\theta$;

and so $$pp_0' = X + Y\theta,$$

where $$X = xx' + xy' + \delta yy', \text{ and } Y = x'y - xy';$$

and so $$PP' = X^2 + XY + \delta Y^2.$$

The identities on which the theorem depends are

$$x(x' + y'\theta) = xx' + xy' + \delta yy' - y'\theta'(x + y\theta),$$

and $$(x' + y')(x + y\theta) = xx' + xy' + \delta yy' + y\theta(x' + y'\theta),$$

the latter being obtained from the former by changing $x$ to $x' + y'$, $x'$ to $x + y$, $y$ to $-y'$, $y'$ to $-y$, and $\theta$ to $\theta'$.

These identities may be written

$$x \,.\, p' \equiv X \text{ (mod. } p),$$

and $$(x' + y') \,.\, p \equiv X \text{ (mod. } p');$$

and therefore $$(p'/p)(p/p') = (x/P)(x' + y'/P')(X/PP').$$

Finally, we obtain the following results, in which for brevity I write

$$(p'/p)(p/p') = (-1)^M.$$

I. $\delta$ even: here $P \equiv x^2 + xy$ (mod. 2), and so $x$ must be odd, and $y$ even, if $P$ is odd; and therefore $p \equiv p' \equiv 1$ (mod. 2). Then

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'-1}{2} + \frac{x-1}{2} \cdot \frac{x'-1}{2}.$$

II. $\delta \equiv 1$ (mod. 4). Here an odd prime $p$ may be $\equiv 1$, or $\theta$, or $1+\theta$ (mod. 2), so that there are six different cases:

(i) $p \equiv p' \equiv 1$ (mod. 2),

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'-1}{2} + \frac{x-1}{2} \cdot \frac{x'-1}{2};$$

(ii) $p \equiv 1$, $p' \equiv \theta$ (mod. 2),

$$M = \frac{x-1}{2} \cdot \frac{x'}{2} + \frac{y}{2} \cdot \frac{y'+1}{2};$$

(iii) $p \equiv p' \equiv \theta$ (mod. 2),

$$M = \frac{x+y+1}{2} \cdot \frac{x'+y'+1}{2} + \frac{y+1}{2} \cdot \frac{y'+1}{2};$$

(iv) $p \equiv 1$, $p' \equiv 1+\theta$ (mod. 2),

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'}{2} + \frac{y}{2} \cdot \frac{y'-1}{2};$$

(v) $p \equiv \theta$, $p' \equiv 1+\theta$ (mod. 2),

$$M = \frac{x+y+1}{2} \cdot \frac{x'+y'}{2} + \frac{x}{2} \cdot \frac{x'+1}{2};$$

(vi) $p \equiv p' \equiv 1+\theta$ (mod. 2),

$$M = \frac{x+1}{2} \cdot \frac{x'+1}{2} + \frac{y+1}{2} \cdot \frac{y'+1}{2} + 1.$$

III. $\delta \equiv 3$ (mod. 4). As before, there are six different cases:

(i) $p \equiv p' \equiv 1$ (mod. 2),

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'-1}{2} + \frac{x-1}{2} \cdot \frac{x'-1}{2};$$

(ii) $p \equiv 1$, $p' \equiv \theta$ (mod. 2),

$$M = \frac{x-1}{2} \cdot \frac{x'+2}{2} + \frac{y}{2} \cdot \frac{y'+1}{2};$$

(iii) $p \equiv p' \equiv \theta$ (mod. 2),

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'-1}{2} + \frac{y-1}{2} \cdot \frac{y'-1}{2};$$

(iv) $p \equiv 1$, $p' \equiv 1+\theta$ (mod. 2),

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'+2}{2} + \frac{y}{2} \cdot \frac{y'-1}{2};$$

(v) $p \equiv \theta$, $p' \equiv 1+\theta$ (mod. 2),

$$M = \frac{x+y-1}{2} \cdot \frac{x'+y'+2}{2} + \frac{x+2}{2} \cdot \frac{x'-1}{2};$$

(vi) $p \equiv p' \equiv 1+\theta$ (mod. 2),

$$M = \frac{x-1}{2} \cdot \frac{x'-1}{2} + \frac{y+1}{2} \cdot \frac{y'+1}{2} + 1.$$

It is remarkable that for all values of $\delta$, when $p \equiv p' \equiv 1$ (mod. 2), the expression for $M$ is the same.

As in the case of systems of the first type, it may be proved that these laws remain true when for $p$ is written $r$, or (and) for $p'$ is written $r'$.

19. Just as in the ordinary Theory of Numbers, the Legendrian symbol was generalised by Jacobi, so the analogous symbol $(h/n)$ may be generalised. In § 11, the symbol was only defined to exist when $n$ is a prime number.

Now let $n = n_1 n_2 n_3 \ldots$, where $n_1, n_2, n_3, \ldots$ are odd prime numbers, all prime to $h$: then we define $(h/n)$ thus

$$(h/n) = (h/n_1)\,(h/n_2)\ldots.$$

It follows at once from this definition that

$$(h/m)\,(h/n) = (h/mn),\ m \text{ and } n \text{ being any odd numbers prime to } h;$$

$$\text{that } (h_1/n)\,(h_2/n)\ldots = (h_1 h_2 \ldots /n),\ h_1,\ h_2 \ldots \text{ being prime to the odd number } n;$$

$$\text{that if } h \equiv k \text{ (mod. } n), \text{ then } (h/n) = (k/n);$$

and lastly, that if $h_0$ is real, $n$ being as before any odd number, whose norm is $N$,

$$(h_0/n) = (h_0/N).$$

Now let $p$ and $p'$ denote any two odd principal numbers in one of the systems which are specially considered in this paper (§ 4); then the prime factors of $p$ or $p'$ are a certain number of odd principal or real primes, together with an even number of odd secondary primes.

Then the laws of reciprocity for the product of the generalised symbols $(p/p')$ and $(p'/p)$ are the same in form as those already proved when $p$ and $p'$ denote odd principal primes. The reason for this is simply that throughout §§ 13—18 no use is made of the supposition that $p$ and $p'$ are primes; all that is assumed about them is that they are odd principal numbers.

20. To complete the Laws of Reciprocity, we must evaluate $(\lambda q/p)\,(p/q)$ for systems of the first type, in which $\lambda^2 = 2$.

Let $p = x' + y'\theta$, $q = x\lambda + y\mu$. Then the following results may be proved:

I. For the systems $d = 5$, 13, 37, for each of which $d \equiv 5$ (mod. 8).

(i) $p \equiv 1$ (mod. 2), then $\quad (\lambda q/p)\,(p/q) = (-1)^{\frac{xy'}{2}}\phi$;

(ii) $p \equiv \theta$, $q \equiv \mu$ (mod. 2), then $\quad (\lambda q/p)\,(p/q) = (-1)^{1+\frac{x+x'}{2}}\phi$;

(iii) $p \equiv \theta$, $q \equiv \lambda + \mu$ (mod. 2), then $\quad (\lambda q/p)\,(p/q) = (-1)^{1+\frac{x-y}{2}}\phi$;

where $$\phi = (-1)^{\frac{1}{8}(x'+y'-1)(x'+y'-3)}.$$

II. For the systems $d = 6$, and 22, for which $d \equiv 6$ (mod. 8).

(i) $p \equiv 1$, $q \equiv \mu$ (mod. 2), then $(\lambda q/p)(p/q) = (-1)^{\frac{x'-1}{2}}\phi$;

(ii) $p \equiv 1$, $q \equiv \lambda + \mu$ (mod. 2), then $(\lambda q/p)(p/q) = (-1)^{\frac{y'}{2}}\phi$;

(iii) $p \equiv 1 + \theta$, $q \equiv \mu$ (mod. 2), then $(\lambda q/p)(p/q) = (-1)^{\frac{x+y+y'}{2}}\phi$;

(iv) $p \equiv 1 + \theta$, $q \equiv \lambda + \mu$ (mod. 2), then $(\lambda q/p)(p/q) = (-1)^{\frac{x-1}{2}}\phi$;

where $$\phi = (-1)^{\frac{x^2-1}{8}}.$$

III. For the systems $d = 10$, and 58, for which $d \equiv 2$ (mod. 8).

(i) $p \equiv 1$ (mod. 2), then $(\lambda q/p)(p/q) = (-1)^{\frac{x(x'+y'-1)}{2}}\phi$;

(ii) $p \equiv 1 + \theta$, $q \equiv \mu$ (mod. 2), then $(\lambda q/p)(p/q) = (-1)^{\frac{x+y+x'+y'-1}{2}}\phi$;

(iii) $p \equiv 1 + \theta$, $q \equiv \lambda + \mu$ (mod. 2), then $(\lambda q/p)(p/q) = (-1)^{\frac{x+x'}{2}}\phi$;

where $$\phi = (-1)^{\frac{x^2-1}{8}}.$$

## BINARY QUADRATIC FORMS.

21. In this branch of the subject the analogy with the ordinary Theory of Numbers is not so complete as in the earlier portions of this paper. As in the ordinary Theory, the quadratic form $ax^2 + bxy + cy^2$ will be denoted by $(a, b, c)$; $a$, $b$, $c$, $x$ and $y$ denote quadratic numbers. A binary quadratic form with quadratic numbers for coefficients will be called throughout the paper briefly a "form." The number $m$ is represented by the form $(a, b, c)$ when $m = ax^2 + bxy + cy^2$, $x$ and $y$ being numbers: it will always be supposed that $x$ is prime to $y$, and that $a$, $b$, and $c$ have no common factor. The determinant of this form is $\Delta = b^2 - 4ac$.

$\Delta$ is therefore always a principal number.

As there is no distinction in the systems here considered between positive and negative numbers, there is nothing corresponding to the division of forms into definite and indefinite forms which occurs in the ordinary Theory.

Forms may be divided into sets, according to whether their coefficients and variables are principal or secondary numbers.

Remembering that a principal and a secondary number may not be added, the following kinds of forms exist:—

(1) $a$, $b$, and $c$ principal numbers.

(2) $a$, $b$, and $c$ secondary numbers; in this and the former case,

either ($\alpha$) $x$ and $y$ principal numbers,

or ($\beta$) $x$ and $y$ secondary numbers.

(3) $a$ and $c$ principal, $b$ secondary.

(4) $a$ and $c$ secondary, $b$ principal; in this and in case (3)

either ($\alpha$) $x$ principal, $y$ secondary,

or ($\beta$) $x$ secondary, $y$ principal.

These will be called the 1st, 2nd, 3rd and 4th *kinds* of forms.

22. A different classification of forms can be made according to the residue of $\Delta$ (mod. 4), since $\Delta \equiv b^2$ (mod. 4).

For the system $D = -20$, $d = 5$, the table in § 10 furnishes the following results:—

(1) $b \equiv 0$ (mod. 2), the form $(a, b, c)$ being of any of the four kinds; then

$$\Delta \equiv 0 \pmod{4}.$$

The form $\left(1,\ 0,\ -\frac{\Delta}{4}\right)$ is an example of this case.

(2) $b \equiv 1 + \theta$ (mod. 2), the form being of the first or fourth kinds; then

$$\Delta \equiv 2\theta \pmod{4}.$$

(3) $b \equiv 1$ (mod. $\lambda$), the form being of the first or fourth kinds; then

$$\Delta \equiv \pm 1 \pmod{4}.$$

(4) $b \equiv \lambda$ (mod. 2), the form being of the second or third kinds; then

$$\Delta \equiv 2 \pmod{4}.$$

(5) $b \equiv \mu$ (mod. $\lambda$), the form being of the second or third kinds; then

$$\Delta \equiv 2 \pm \theta \pmod{4}.$$

And conversely for any value of $\Delta$ satisfying one of the above congruences forms exist of the corresponding kinds; for $ac = \frac{1}{4}(b^2 - \Delta)$; the number on the right can always be split into the principal factors 1 and $\frac{1}{4}(b^2 - \Delta)$, and it is fairly evident that out of the infinity of possible values of $b$, $\frac{1}{4}(b^2 - \Delta)$ will often split into the product of two secondary factors.

23. Forms are said to be equivalent when the substitution

$$\begin{cases} x = \alpha x' + \beta y', \\ y = \gamma x' + \delta y', \end{cases} \text{ where } \alpha\delta - \beta\gamma = 1$$

transforms one of them into the other.

Then the inverse substitution

$$\begin{cases} x' = \delta x - \beta y, \\ y' = -\gamma x + \alpha y \end{cases}$$

transforms the latter into the former, so that the relation of equivalence is a mutual one.

There are four "kinds" of such substitutions:

(i) $\alpha$, $\beta$, $\gamma$, $\delta$ all principal; then $x$ and $y$ are both of the same kind, i.e. both principal, or both secondary.

(ii) $\alpha$, $\beta$, $\gamma$, $\delta$ all secondary; $x$ and $y$ are both of the same kind.

(iii) $\alpha$ and $\delta$ principal, $\beta$ and $\gamma$ secondary; $x$ and $y$ are of different kinds.

(iv) $\alpha$ and $\delta$ secondary, $\beta$ and $\gamma$ principal; $x$ and $y$ are of different kinds.

The first and second kinds of substitutions alone are applicable to forms of the first or second kind; and the third and fourth kinds of substitutions alone to forms of the third or fourth kind.

If the form $(a', b', c')$ is equivalent to $(a, b, c)$, being obtained from the latter by the substitution $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then

$$a' = a\alpha^2 + b\alpha\gamma + c\gamma^2,$$
$$b' = 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta,$$
$$c' = a\beta^2 + b\beta\delta + c\delta^2.$$

Therefore a form is necessarily transformed into another of the same kind.

Since $b'^2 - 4a'c' = (\alpha\delta - \beta\gamma)^2(b^2 - 4ac) = b^2 - 4ac$, equivalent forms have the same determinant.

Let
$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad S' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix},$$

then the substitution arising from their composition in this order is

$$SS' = \begin{pmatrix} \alpha\alpha' + \beta\gamma' & \alpha\beta' + \beta\delta' \\ \gamma\alpha' + \delta\gamma' & \gamma\beta' + \delta\delta' \end{pmatrix}.$$

Now let $\sigma_\kappa$ denote a substitution of the $\kappa$th kind.

Then $\sigma_1^2$ and $\sigma_2^2$ are of the first kind,

$\sigma_1\sigma_2$ and $\sigma_2\sigma_1$ are of the second kind.

Therefore the substitutions of the first and second kinds together form a group, of which those of the first kind form a sub-group. So also the substitutions of the third and fourth kinds form a group, of which those of the third kind form a sub-group. Other compositions of substitutions, e.g. $\sigma_1\sigma_3$, are impossible, for they would necessitate the addition of principal and secondary numbers.

We can now give a more precise definition of "equivalence"; it is an essential part of the notion of equivalence, that forms that are equivalent to the same form should be equivalent to one another.

In other words the set of substitutions, assumed to exist by the definition of "equivalence," must form a group.

There are accordingly two kinds of equivalence, corresponding to the two groups of substitutions applicable to a given form. A form of the first or second kind will be called "narrowly equivalent" to any form obtained from it by a substitution of the first kind, but "widely equivalent" to a form obtained from it by a substitution of either the first or the second kind. Similar definitions of the terms "narrow" and "wide" equivalence apply to forms of the third and fourth kinds.

Forms which are equivalent are said to belong to the same class, the class being narrow or wide, according to whether the equivalence is narrow or wide.

A wide class obviously contains the whole of a narrow class, if it contains a single form belonging to the narrow class.

Since all forms of a class are of the same kind, we can speak of the "kind" of the class.

24. There is, as in the ordinary Theory, a close connection between any class of forms and the set of numbers representable by any form of the class.

For, if $a' = a\alpha^2 + b\alpha\gamma + c\gamma^2$, $\alpha$ being prime to $\gamma$, and $\beta$ and $\delta$ be chosen so that $\alpha\delta - \beta\gamma = 1$ (§ 8), then the substitution $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ converts the form $(a, b, c)$ into an equivalent form $(a', b', c')$.

Conversely, if $(a', b', c')$ is equivalent to $(a, b, c)$, then the extreme coefficients $a'$ and $c'$ are representable by $(a, b, c)$. Hence the following theorems:—

The set of the extreme coefficients of all the forms of a narrow class of the first or second kind coincides with the set of all numbers representable by a form of the class, the variables $x$ and $y$ taking coprime principal values.

The set of the extreme coefficients of all the forms of a wide class of the first or second kind coincides with the set of all numbers representable by a form of the class, the variables taking all possible coprime values.

The set of first coefficients of all the forms of a narrow class of the third or fourth kind coincides with the set of all numbers representable by a form of the class, $x$ taking principal and $y$ taking secondary values; and the set of third coefficients of the same to the set of numbers representable by a form of the class, $x$ taking secondary and $y$ principal coprime values.

Lastly, the set of the extreme coefficients of all the forms of a wide class of the third and fourth kind coincides with the set of all numbers representable by a form of the class, the variables taking all possible coprime values.

If $a'$ is an odd number representable by $(a, b, c)$, and $b'$, $c'$ have the same meanings as before, then

$$b'^2 - 4a'c' = \Delta,$$

and so

$$b'^2 \equiv \Delta \pmod{a'}.$$

Therefore, if $a'$ is representable by a form of the first or fourth kind (in each of which the middle coefficient is a principal number), $a'$ must satisfy the condition $(\Delta/a') = 1$.

And if $a'$ is representable by a form of the second or third kind, the condition is $(\Delta/a') = 1$, or $-1$, according to the value of $a'$ and the system considered. (§ 11 (ii).)

Conversely, if $b'^2 \equiv \Delta \pmod{a'}$ is soluble, then writing $\dfrac{b'^2 - \Delta}{a'} = c'$, the form $(a', b', c')$ is of determinant $\Delta$, and represents the number $a'$.

25. In the ordinary Theory of Numbers, there is an elementary proof of the finiteness of the number of classes for a given determinant, which depends on the method of Reduction. This method of proof does not apply to the systems of numbers now being studied.

It will be remembered that in the ordinary Theory the first step in the process is to apply to the form $(a, b, c)$ the substitution $\begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}$, thus producing the form $(c, b', a')$, where

$$b' = -b - 2c\delta,$$
$$a' = a + b\delta + c\delta^2;$$

and $\delta$ is then determined so that $|b'| \gtreqless c$.

In this way a reduced form $(A, B, C)$ is obtained, such that

$$C \nless A \nless |B|.$$

Now, in any of the systems of numbers here considered the substitution $\begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}$ is either of the first or fourth kind, according as $\delta$ is principal or secondary. In either case, it is not always possible to find a residue $b'$ of $-b$ to mod. $2c$ such that $N(b') < N(2c)$, and the process of reduction therefore breaks down. If this were possible when $b$ and $c$ are any principal numbers, the Euclidean process for finding their greatest common factor would work, and there would be no need of secondary numbers to complete the laws of factorisation. And it is easy to prove the impossibility of always finding a residue $b'$ such that $N(b') < N(2c)$, where either $b$ or $c$, or both, are secondary numbers. Possibly Dirichlet's analytical method of determining the class-number for a given determinant in the ordinary Theory would apply to these systems of quadratic numbers.

## GENERIC CHARACTERS.

26. Just as in the ordinary Theory of Numbers, the classes for a given determinant may be divided into genera. Since the results mainly depend on the residues of squares to moduli consisting of powers of 2 or its factors, I shall confine the remainder of this paper to the case $D = -20$, i.e. $d = 5$; and for brevity I shall only consider forms of the first kind.

All the results of this nature are derived from the simple identity

$$4nn' = x^2 - \Delta y^2,$$

where

$$n = au^2 + buv + cv^2,$$
$$n' = au'^2 + bu'v' + cv'^2,$$
$$x = 2auu' + b(uv' + u'v) + 2cvv',$$
$$y = uv' - u'v,$$
$$\Delta = b^2 - 4ac.$$

Three cases will need separate consideration, according as $b$ is even, semi-even, or odd, i.e. according as $\Delta \equiv 0$, $2\theta$, or $\pm 1$ (mod. 4). (§ 22.)

(1) When $\Delta \equiv 0 \pmod{4} = 4\delta$, the above identity may be divided by 4, giving

$$nn' = x^2 - \delta y^2,$$

$x$ here meaning half the expression above given for $x$.

(2) When $\Delta \equiv 2\theta \pmod{4} = 2\delta$,

$$2nn' = x^2 - \delta y^2,$$

where

$$\lambda x = 2auu' + b(uv' + u'v) + 2cvv'.$$

(3) When $\Delta \equiv \pm 1 \pmod{4}$,

$$4nn' = x^2 - \Delta y^2.$$

With regard to narrow classes, in any of these three cases, there exist quadratic characters precisely analogous to the quadratic characters in the ordinary theory.

For if $t$ be any odd prime factor of $\Delta$, we have $4nn' \equiv x^2 \pmod{t}$, where $x$ is principal (since the form is of the first kind, and the numbers $u$, $u'$, $v$, $v'$ are principal).

Therefore $$(nn'/t) = +1,$$

that is $$(n/t) = (n'/t).$$

27. There exist besides supplementary characters, which depend on theorems as to the residues of $n$ and $n'$ (being odd numbers) to moduli of the form $\lambda^k$.

These characters may be defined as follows: $n = n_0 + n_1\theta$ is an odd number whose norm is $N$; then

$$\psi(n) = (-1)^{n_1},$$
$$\chi(n) = (-1)^{\frac{N-1}{4}},$$
$$\omega(n) = (-1)^{\frac{1}{8}(n_0+\psi n_1-1)(n_0+\psi n_1-3)}.$$

Where no ambiguity is caused, I shall write $\psi$ for $\psi(n)$, etc.

Therefore $\psi = +1$ or $-1$, according as $n \equiv 1$ or $\theta \pmod{2}$;

when $\chi = +1$, $\quad n \equiv \pm 1$, or $\pm(2+\theta) \pmod{4}$,

when $\chi = -1$, $\quad n \equiv \pm(1+2\theta)$, or $\pm\theta \pmod{4}$.

And the following table is easily deduced from the definitions just given.

| Residues of $n$ to modulus $\lambda^5$, i.e. $4\lambda$. | $\psi$ | $\chi$ | $\omega$ |
|---|---|---|---|
| $+1,\quad +3$ | + | + | + |
| $-1,\quad -3$ | + | + | − |
| $1+2\theta,\quad -1+2\theta$ | + | − | + |
| $-1-2\theta,\quad 1-2\theta$ | + | − | − |
| $2+\theta,\quad 2-\theta$ | − | + | + |
| $-2-\theta,\quad -2+\theta$ | − | + | − |
| $-\theta,\quad 4+\theta$ | − | − | + |
| $\theta,\quad 4-\theta$ | − | − | − |

From the definitions or from this table it may be verified that

$$\psi(nn') = \psi(n)\psi(n'), \quad \chi(nn') = \chi(n)\chi(n'), \quad \omega(nn') = \omega(n)\omega(n').$$

One specimen of the reasoning by which the existence of these supplementary characters is proved will be sufficient, and I shall then present a table shewing all the appropriate supplementary characters of narrow classes for the various values of $\Delta$.

In the case $\qquad \Delta = 4\delta$, and $\delta \equiv \pm(1+\theta)$ (mod. $4\lambda$).

Then $\qquad nn' \equiv x^2 \pm (1+\theta)y^2$ (mod. $4\lambda$),

where $x$ must be odd, but $y$ may be odd, semi-even, or even, both $x$ and $y$ being principal; then (§ 10, table)

$$x^2 \equiv 1 \text{ or } 3 \text{ (mod. } 4\lambda),$$

$$y^2 \equiv \pm 1,\ 2\theta \text{ or } 0 \text{ (mod. } 4),$$

and so $\qquad nn' \equiv 2+\theta,\ -\theta,\ 4+\theta,\ 2-\theta,\ -1+2\theta,\ 1+2\theta,\ 1 \text{ or } 3$ (mod. $4\lambda$).

Therefore $\qquad \omega(nn') = +1,$

that is $\qquad \omega(n) = \omega(n').$

## SUPPLEMENTARY CHARACTERS OF NARROW CLASSES.

I. $\Delta = 4\delta$.

| Odd Residues of $\delta$ to mod. $4\lambda$. | Characters. | Even Residues of $\delta$ to mod. $4\lambda$. | Characters. |
|---|---|---|---|
| $\pm 1$ | $\psi$ | $0$ | $\psi, \chi, \omega$ |
| $\pm 3$ | $\psi$ | $4$ | $\psi, \chi$ |
| $\pm(1+2\theta)$ | $\psi$ | $\pm 2$ | $\psi, \chi$ |
| $\pm(1-2\theta)$ | $\psi$ | $\pm 2\theta$ | $\psi$ |
| $\pm\theta$ | $\psi\chi$ | $2+2\theta$ | $\psi, \chi\omega$ |
| $\pm(4+\theta)$ | $\psi\chi$ | $2-2\theta$ | $\psi, \omega$ |
| $\pm(2+\theta)$ | $\chi$ | $\pm(1+\theta)$ | $\omega$ |
| $\pm(2-\theta)$ | $\chi$ | $\pm(1-\theta)$ | $\chi\omega$ |
| | | $\pm(3+\theta)$ | $\psi\chi\omega$ |
| | | $\pm(3-\theta)$ | $\psi\omega$ |

II. $\Delta = 2\delta \equiv 2\theta \pmod{4}$.

| Residues of $\delta$ to mod. 4. | Characters. |
|---|---|
| $\pm\theta$ | $\psi$ |
| $\pm(2+\theta)$ | none |

III. $\Delta \equiv \pm 1 \pmod{4}$; there are no supplementary characters in this case.

28. Two classes of a given determinant are said to belong to the same "genus," when all their generic characters have the same values.

Half the assignable genera of narrow classes of the first kind for a given determinant are impossible; this result is obtained (as in the ordinary Theory) by applying the Law of Quadratic Reciprocity to the equation $(\Delta/n) = +1$ (§ 24), $n$ being here any odd number prime to $\Delta$ representable by some form of the first kind of determinant $\Delta$.

For example, let $\Delta = 4ts^2$, $s$ being principal and the largest square in $\Delta$, and $t$ also principal and $= t_1 t_2 \ldots$; and suppose $t \equiv \pm\theta \pmod{4}$. Then $n$ being $n_0 + n_1\theta$, the law of reciprocity (§ 15) gives

$$(\Delta/n) = (t/n) = \epsilon\,(n/t),$$

where $\epsilon = (-1)^{\frac{n_0}{2}}$, if $n_1$ is odd, but $(-1)^{\frac{n_1}{2}}$, if $n_1$ is even.

That is, $\epsilon = +1$, if $n \equiv \pm 1$ or $\pm\theta$ (mod. 4),

and $\epsilon = -1$, if $n \equiv \pm(1+2\theta)$ or $\pm(2+\theta)$ (mod. 4).

Therefore $\epsilon = \psi\chi$; and $(n/t) = (n/t_1)(n/t_2)\ldots\ldots$,

so that the condition $(\Delta/n) = +1$

becomes $\psi\chi \,.\, (n/t_1) \,.\, (n/t_2) \ldots\ldots = +1.$

Similar applications of the law of reciprocity (§§ 15 and 20) to the numerous cases furnish the facts set out in the table below, which is arranged in a similar manner to Dirichlet's table in the ordinary Theory (Mathews, *Theory of Numbers*, Pt. I. p. 135), and to H. J. S. Smith's table in his paper "On Complex Binary Quadratic Forms" (in the system of numbers $x + y\sqrt{-1}$) (*Collected Papers*, Vol. I. p. 421). In the table $s^2$ denotes the largest square dividing $\delta$ in cases (I) and (II), or dividing $\Delta$ in case (III); $t_1, t_2 \ldots\ldots$ are the different odd prime factors of $t$ which is itself odd; $s_1, s_2 \ldots\ldots$ are those odd prime factors of $s$ which do not divide $t$; and $I$ is the index of the highest power of $\lambda$ contained in $s$. In each line is the complete set of characters for the corresponding value of $\Delta$, those characters to the left of the vertical line being subject to the condition that their product is $+1$.

## Possible Generic Characters for Narrow Classes.

### I. $\Delta = 4\delta$.

(1) $\delta = ts^2$ $\begin{cases} s \text{ principal, and } t \equiv \pm 1 \text{ (mod. 4)}, \\ \text{or } s \text{ secondary, and } t \equiv \pm(2+\theta) \text{ (mod. 4)}, \end{cases}$

| | | |
|---|---|---|
| $I = 0$ or $1$ | $(n/t_1), \ldots\ldots$ | $\psi, (n/s_1), \ldots\ldots$ |
| $I = 2$ | $(n/t_1), \ldots\ldots$ | $\psi, \chi, (n/s_1), \ldots\ldots$ |
| $I > 2$ | $(n/t_1), \ldots\ldots$ | $\psi, \chi, \omega, (n/s_1), \ldots\ldots$ |

(2) $\delta = ts^2$ $\begin{cases} s \text{ principal, and } t \equiv \pm(1+2\theta) \text{ (mod. 4)}, \\ \text{or } s \text{ secondary, and } t \equiv \pm\theta \text{ (mod. 4)}, \end{cases}$

| | | |
|---|---|---|
| $I = 0$ or $1$ | $\psi, (n/t_1), \ldots\ldots$ | $(n/s_1), \ldots\ldots$ |
| $I = 2$ | $\psi, (n/t_1), \ldots\ldots$ | $\chi, (n/s_1), \ldots\ldots$ |
| $I > 2$ | $\psi, (n/t_1), \ldots\ldots$ | $\chi, \omega, (n/s_1), \ldots\ldots$ |

(3) $\delta = ts^2$ $\begin{cases} s \text{ principal, and } t \equiv \pm\theta \text{ (mod. 4)}, \\ \text{or } s \text{ secondary, and } t \equiv \pm(1+2\theta) \text{ (mod. 4)}, \end{cases}$

| | | |
|---|---|---|
| $I = 0$ | $\psi\chi, (n/t_1), \ldots\ldots$ | $(n/s_1), \ldots\ldots$ |
| $I = 1$ or $2$ | $\psi, \chi, (n/t_1), \ldots\ldots$ | $(n/s_1), \ldots\ldots$ |
| $I > 2$ | $\psi, \chi, (n/t_1), \ldots\ldots$ | $\omega, (n/s_1), \ldots\ldots$ |

(4) $\delta = ts^2$ $\begin{cases} s \text{ principal, and } t \equiv \pm(2+\theta) \text{ (mod. 4)}, \\ \text{or } s \text{ secondary, and } t \equiv \pm 1 \text{ (mod. 4)}, \end{cases}$

| | | |
|---|---|---|
| $I = 0$ | $\chi, (n/t_1), \ldots\ldots$ | $(n/s_1), \ldots\ldots$ |
| $I = 1$ or $2$ | $\chi, (n/t_1), \ldots\ldots$ | $\psi, (n/s_1), \ldots\ldots$ |
| $I > 2$ | $\chi, (n/t_1), \ldots\ldots$ | $\psi, \omega, (n/s_1), \ldots\ldots$ |

(5) $\delta = \lambda t s^2$ $\begin{cases} s \text{ principal, and } \lambda t \equiv \pm(1+\theta) \pmod{4\lambda}, \\ \text{or } s \text{ secondary, and } \lambda t \equiv \pm(1-\theta) \pmod{4\lambda}, \end{cases}$

| | | |
|---|---|---|
| $I = 0$ | $\omega$, $(n/t_1)$, ..... | $(n/s_1)$, ..... |
| $I = 1$ | $\omega$, $(n/t_1)$, ..... | $\psi$, $(n/s_1)$, ..... |
| $I > 1$ | $\omega$, $(n/t_1)$, ..... | $\psi$, $\chi$, $(n/s_1)$, ..... |

(6) $\delta = \lambda t s^2$ $\begin{cases} s \text{ principal, and } \lambda t \equiv \pm(1-\theta) \pmod{4\lambda}, \\ \text{or } s \text{ secondary, and } \lambda t \equiv \pm(1+\theta) \pmod{4\lambda}, \end{cases}$

| | | |
|---|---|---|
| $I = 0$ | $\chi\omega$, $(n/t_1)$, ..... | $(n/s_1)$, ..... |
| $I = 1$ | $\chi\omega$, $(n/t_1)$, ..... | $\psi$, $(n/s_1)$, ..... |
| $I > 1$ | $\chi$, $\omega$, $(n/t_1)$, ..... | $\psi$, $(n/s_1)$, ..... |

(7) $\delta = \lambda t s^2$ $\begin{cases} s \text{ principal, and } \lambda t \equiv \pm(3+\theta) \pmod{4\lambda}, \\ \text{or } s \text{ secondary, and } \lambda t \equiv \pm(3-\theta) \pmod{4\lambda}, \end{cases}$

| | | |
|---|---|---|
| $I = 0$ | $\psi\chi\omega$, $(n/t_1)$, ..... | $(n/s_1)$, ..... |
| $I = 1$ | $\psi$, $\chi\omega$, $(n/t_1)$, ..... | $(n/s_1)$, ..... |
| $I > 1$ | $\psi$, $\chi$, $\omega$, $(n/t_1)$, ..... | $(n/s_1)$, ..... |

(8) $\delta = \lambda t s^2$ $\begin{cases} s \text{ principal, and } \lambda t \equiv \pm(3-\theta) \pmod{4\lambda}, \\ \text{or } s \text{ secondary, and } \lambda t \equiv \pm(3+\theta) \pmod{4\lambda}, \end{cases}$

| | | |
|---|---|---|
| $I = 0$ | $\psi\omega$, $(n/t_1)$, ..... | $(n/s_1)$, ..... |
| $I = 1$ | $\psi$, $\omega$, $(n/t_1)$, ..... | $(n/s_1)$, ..... |
| $I > 1$ | $\psi$, $\omega$, $(n/t_1)$, ..... | $\chi$, $(n/s_1)$, ..... |

II. $\Delta \equiv 2\theta \pmod{4} = 2\delta = 2ts^2$.

| | | |
|---|---|---|
| $\delta \equiv \pm\theta \pmod{4}$ | $\psi$, $(n/t_1)$, ..... | $(n/s_1)$, ..... |
| $\delta \equiv \pm(2+\theta) \pmod{4}$ | $(n/t_1)$, ..... | $(n/s_1)$, ..... |

III. $\Delta \equiv \pm 1 \pmod{4} = ts^2$.

| | |
|---|---|
| $(n/t_1)$, ..... | $(n/s_1)$, ..... |

29. The generic characters of wide classes remain to be considered; we shall find that characters here occur of a kind which have no analogy in the ordinary Theory. Using the notation of § 26, if $n$ is a number of the wide class represented by the form $(a, b, c)$ of the first kind, $u$ and $v$ are either both principal or both secondary (§ 24). Similarly for $u'$ and $v'$; and so $x$ and $y$ are both principal, or both secondary. Let $t$ be any odd prime factor of $\Delta$.

Then $$4nn' \equiv x^2 \pmod{t}.$$

If $x$ is principal, this gives $$(nn'/t) = +1.$$

If $x$ is secondary $$(nn'/t) \equiv x^{T-1} \equiv (-1)^{\frac{T^2-1}{8}} \pmod{t}. \quad (\S\ 9,\ \text{II.})$$

Now either $T \equiv \pm 1 \pmod{8}$, in which case $(nn'/t) = +1$, whether $x$ is principal or secondary, and we get the quadratic character $(n/t)$; or $T \equiv \pm 3 \pmod{8}$, in which case

$(nn'/t)$ is $+1$ or $-1$, according as $x$ is principal or secondary. Let $t_1'$ and $t_2'$ be two odd prime factors of $\Delta$, both of whose norms are $\equiv \pm 3 \pmod{8}$; then it follows that $(nn'/t_1't_2') = +1$, whether $x$ is principal or secondary; that is, $(n/t_1't_2') = (n'/t_1't_2')$; thus in the case of wide classes, there is a quadratic character corresponding to every pair of those odd prime factors of $\Delta$, whose norms are $\equiv \pm 3 \pmod{8}$. These however are not all independent, but it is evident that if $t_1'$, $t_2'$, ..... $t_r'$ are the prime factors of $\Delta$ of this kind, then $(n/t_1't_2')$, $(n/t_1't_3')$, ..... $(n/t_1't_r')$ are all independent, and form a complete set, $r-1$ in number; for

$$(n/t_1't_l') \times (n/t_1't_m') = (n/t_l't_m').$$

For certain values of $\Delta$ the supplementary characters $\chi$, $\omega$, etc. occur; and there are also in some cases mixed characters, as $\psi\,(n/t_1')$, etc.

The latter arises when $\psi(nn') = +1$, and $(nn'/t_1') = +1$, if $x$ is principal; but $\psi(nn') = -1$, $(nn'/t_1') = -1$, if $x$ is secondary. Then $\psi(nn')\,(nn'/t_1') = +1$, in either case: and so $\psi(n).(n/t_1') = \psi(n').(n'/t_1')$.

I now set out in a table the supplementary and mixed characters of wide classes for the various values of $\Delta$; it will be observed that they are identical with the corresponding results for narrow classes, except that some of the supplementary characters are associated with $(n/t_1')$.

SUPPLEMENTARY AND MIXED CHARACTERS OF WIDE CLASSES.

I. $\Delta = 4\delta$.

| Odd Residues of $\delta$ to mod. $4\lambda$. | Characters. | Even Residues of $\delta$ to mod. $4\lambda$. | Characters. |
|---|---|---|---|
| $\pm 1$ | $\psi\,(n/t_1')$ | $0$ | $\psi\,(n/t_1')$, $\chi$, $\psi\omega$ |
| $\pm 3$ | $\psi\,(n/t_1')$ | $4$ | $\psi\,(n/t_1')$, $\chi$ |
| $\pm(1+2\theta)$ | $\psi\,(n/t_1')$ | $\pm 2$ | $\psi\,(n/t_1')$, $\chi$ |
| $\pm(1-2\theta)$ | $\psi\,(n/t_1')$ | $\pm 2\theta$ | $\psi\,(n/t_1')$ |
| $\pm\theta$ | $\psi\chi\,(n/t_1')$ | $2+2\theta$ | $\psi\,(n/t_1')$, $\psi\chi\omega$ |
| $\pm(4+\theta)$ | $\psi\chi\,(n/t_1')$ | $2-2\theta$ | $\psi\,(n/t_1')$, $\psi\omega$ |
| $\pm(2+\theta)$ | $\chi$ | $\pm(1+\theta)$ | $\omega\,(n/t_1')$ |
| $\pm(2-\theta)$ | $\chi$ | $\pm(1-\theta)$ | $\chi\omega\,(n/t_1')$ |
| | | $\pm(3+\theta)$ | $\psi\chi\omega$ |
| | | $\pm(3-\theta)$ | $\psi\omega$ |

II. $\Delta = 2\delta \equiv 2\theta \pmod{4}$.

| Residues of $\delta$ to mod. 4. | Characters. |
|---|---|
| $\pm\theta$ | $\psi\,(n/t_1')$ |
| $\pm(2+\theta)$ | none |

III. $\Delta \equiv \pm 1 \pmod{4}$. No supplementary, or mixed characters.

30. In the following table, the notation is the same as in the corresponding table for narrow classes, the only difference being that the dashed letters $t_1'$, $t_2'$, ... $s_1'$, ... denote prime factors of $t$ and $s$ whose norms are $\equiv \pm 3 \pmod{8}$, the other factors being denoted by undashed letters: and $r$ denotes the number of the former kind of factors of $t$. It is evident that if $T \equiv \pm 1 \pmod{8}$, then $r$ is even, and if $T \equiv \pm 3 \pmod{8}$, then $r$ is odd; and it is easy to prove that if

$$t \equiv \pm 1,\ \pm(2+\theta),\ \pm(\lambda+\mu),\ \text{or}\ \pm(2\lambda-\mu) \pmod{4},$$

then
$$T \equiv \pm 1 \pmod{8};$$

while if
$$t \equiv \pm(1+2\theta),\ \pm\theta,\ \pm\mu,\ \text{or}\ \pm(\lambda-\mu) \pmod{4},$$

then
$$T \equiv \pm 3 \pmod{8}.$$

Bearing these results in mind as to $r$ being even or odd, it will be seen that the product of the generic characters to the left of the vertical line is the same as in the previous table; this is of course necessary, since the product in question is a transformation of $(\Delta/n)$, and the latter expression is equal to $+1$, for either narrow or wide classes.

In the event of there being no factors of the form $t_1'$ or $s_1'$, or only one of them, the results given below need some modification; it is however easy to see in each such case what the complete set of characters is. For instance in the first line of the table, if there is no $t_1'$, but there is $s_1'$, ..., the characters are

$$(n/t_1),\ (n/t_2),\ \ldots\ldots \mid \psi\,.\,(n/s_1'),\ (n/s_1),\ \ldots\ldots\ (n/s_1's_2'),\ \ldots\ldots;$$

but if $s_1'$ is also absent, then the mixed character containing $\psi$ disappears, and the characters are

$$(n/t_1),\ (n/t_2),\ \ldots\ldots \mid (n/s_1),\ \ldots\ldots.$$

## POSSIBLE GENERIC CHARACTERS FOR WIDE CLASSES.

### I. $\Delta = 4\delta$.

(1) $\delta = ts^2 \left\{\begin{array}{l} s \text{ principal, and } t \equiv \pm 1 \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm(2+\theta) \pmod{4} \end{array}\right\}$ $r$ even.

| | | |
|---|---|---|
| $I = 0$ or $1$ | $(n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\psi(n/t_1'), (n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I = 2$ | $(n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\psi(n/t_1'), \chi, (n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I > 2$ | $(n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\psi(n/t_1'), \chi, \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots$ |

(2) $\delta = ts^2 \left\{\begin{array}{l} s \text{ principal, and } t \equiv \pm(1+2\theta) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm\theta \pmod{4} \end{array}\right\}$ $r$ odd.

| | | |
|---|---|---|
| $I = 0$ or $1$ | $\psi(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $(n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I = 2$ | $\psi(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\chi, (n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I > 2$ | $\psi(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\chi, \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots.$ |

(3) $\delta = ts^2 \left\{\begin{array}{l} s \text{ principal, and } t \equiv \pm\theta \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm(1+2\theta) \pmod{4} \end{array}\right\}$ $r$ odd.

| | | |
|---|---|---|
| $I = 0$ | $\psi\chi(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $(n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I = 1$ or $2$ | $\psi\chi(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\chi, (n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I > 2$ | $\psi\chi(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\chi, \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots$ |

(4) $\delta = ts^2 \left\{\begin{array}{l} s \text{ principal, and } t \equiv \pm(2+\theta) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm 1 \pmod{4} \end{array}\right\}$ $r$ even.

| | | |
|---|---|---|
| $I = 0$ | $\chi, (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $(n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I = 1$ or $2$ | $\chi, (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\psi(n/t_1'), (n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I > 2$ | $\chi, (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\psi(n/t_1'), \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots.$ |

(5) $\delta = \lambda ts^2 \left\{\begin{array}{l} s \text{ principal, and } t \equiv \pm\mu \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm(\lambda-\mu) \pmod{4} \end{array}\right\}$ $r$ odd.

| | | |
|---|---|---|
| $I = 0$ | $\omega(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $(n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I = 1$ | $\omega(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots$ |
| $I > 1$ | $\omega(n/t_1'), (n/t_1), \ldots, (n/t_1't_2'), \ldots, (n/t_1't_r')$ | $\chi, \psi\omega, (n/s_1), \ldots, (n/t_1's_1'), \ldots.$ |

(6) $\delta = \lambda t s^2 \begin{cases} s \text{ principal, and } t \equiv \pm(\lambda - \mu) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm \mu \pmod{4} \end{cases} \Bigg\} \; r \text{ odd.}$

| | | |
|---|---|---|
| $I = 0$ | $\chi\omega\,(n/t_1'),\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $(n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots$ |
| $I = 1$ | $\chi\omega\,(n/t_1'),\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $\psi\chi\omega,\ (n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots$ |
| $I > 1$ | $\chi,\ \omega\,(n/t_1'),\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $\psi\omega,\ (n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots.$ |

(7) $\delta = \lambda t s^2 \begin{cases} s \text{ principal, and } t \equiv \pm(\lambda + \mu) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm(2\lambda - \mu) \pmod{4} \end{cases} \Bigg\} \; r \text{ even.}$

| | | |
|---|---|---|
| $I = 0$ | $\psi\chi\omega,\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $(n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots$ |
| $I = 1$ | $\psi\chi\omega,\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $\psi\,(n/t_1'),\ (n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots$ |
| $I > 1$ | $\chi,\ \psi\omega,\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $\psi\,(n/t_1'),\ (n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots.$ |

(8) $\delta = \lambda t s^2 \begin{cases} s \text{ principal, and } t \equiv \pm(2\lambda - \mu) \pmod{4} \\ \text{or } s \text{ secondary, and } t \equiv \pm(\lambda + \mu) \pmod{4} \end{cases} \Bigg\} \; r \text{ even.}$

| | | |
|---|---|---|
| $I = 0$ | $\psi\omega,\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $(n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots$ |
| $I = 1$ | $\psi\omega,\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $\psi\,(n/t_1'),\ (n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots$ |
| $I > 1$ | $\psi\omega,\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $\psi\,(n/t_1'),\ \chi,\ (n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots.$ |

II. $\Delta \equiv 2\theta \pmod{4} = 2\delta = 2ts^2.$

| | | |
|---|---|---|
| $\delta \equiv \pm\theta \pmod{4}$, $r$ odd | $\psi\,(n/t_1'),\ (n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $(n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots$ |
| $\delta \equiv \pm(2+\theta) \pmod{4}$, $r$ even | $(n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $(n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots$ |

III. $\Delta \equiv \pm 1 \pmod{4} = ts^2$; $r$ even.

| | |
|---|---|
| $(n/t_1),\ \ldots,\ (n/t_1't_2'),\ \ldots,\ (n/t_1't_r')$ | $(n/s_1),\ \ldots,\ (n/t_1's_1'),\ \ldots.$ |

31. As an illustration of the results obtained in reference to Binary Quadratic Forms, I now consider the case $\Delta = -4$. Let $n$ be an odd principal number, representable by a form of the first kind having this determinant; then (§ 24), $n_1$ being any prime factor of $n$,

$$(-4/n_1) = +1.$$

Therefore $(-1/N_1) = +1$, that is $N_1 \equiv 1 \pmod{4}$. But the norm of any odd secondary number $\equiv 3 \pmod{4}$, so $n_1$ and therefore every prime factor of $n$ is either a principal or a real prime. And conversely all such primes and all numbers composed of them are representable by some form of the first kind with this determinant.

I have searched for forms of the first kind in the same way as is done in the ordinary theory, and after eliminating all forms (within the limit of my search) narrowly equivalent to simpler ones, the following forms remain:—$(1,\ 0,\ 1)$, $(\theta,\ 4,\ -\theta)$, $(\theta,\ -4,\ -\theta)$.

The form $(-1, 0, -1)$ is equivalent to $(1, 0, 1)$, the former being derived from the latter by the substitution

$$\begin{pmatrix} 2, & \theta \\ \theta, & -2 \end{pmatrix}.$$

First consider narrow classes; then the first line of the Table I. (§ 28) for $\Delta = 4\delta$ shews that there is one generic character $\psi$, which apparently can be either $+1$ or $-1$; this is an exception to the general rule that half the assignable genera are impossible, this exception being due to the fact that when $\Delta = -4$, there is no generic character $(n/t_1)$, $t_1$ not existing in this case.

Both these genera in fact exist, the class represented by the form $(1, 0, 1)$ having the character $\psi = +1$, and the classes represented by $(\theta, \pm 4, -\theta)$ having the character $\psi = -1$; neither of the latter classes can therefore be narrowly equivalent to $(1, 0, 1)$.

Further, it is not difficult to prove that the forms $(\theta, 4, -\theta)$ and $(\theta, -4, -\theta)$ are non-equivalent (narrowly). I have accordingly proved the existence of at least three narrow classes of forms of the first kind; there may be more such, for, as pointed out in § 25, I am not aware of any method of ascertaining the number of classes of a given determinant.

Turning now to wide equivalence, the three forms above mentioned are all equivalent; for

$$\theta = \lambda^2 + \mu^2,$$

$$2 = \lambda\mu + \lambda(\lambda - \mu),$$

$$-\theta = \lambda^2 + (\lambda - \mu)^2,$$

and therefore the substitutions $\begin{pmatrix} \mu, & \lambda \\ \lambda, & \lambda - \mu \end{pmatrix}$ and $\begin{pmatrix} -\mu, & \lambda \\ \lambda, & -\lambda + \mu \end{pmatrix}$ respectively convert $(1, 0, 1)$ into $(\theta, 4, -\theta)$ and $(\theta, -4, -\theta)$. There is apparently therefore only one wide class; it has no generic character (§ 30).

TABLE I.

32. Systems of numbers of the first type; $D=-4d$, $\lambda=\sqrt{2}$.

| $-D$ | $\theta=\sqrt{-d}$ | $\mu$ | $\lambda\mu$ | $\mu^2$ | $\lambda\theta$ | $\mu\theta$ |
|---|---|---|---|---|---|---|
| 20 | $\sqrt{-5}$ | $\frac{1+\sqrt{-5}}{\sqrt{2}}$ | $1+\theta$ | $-2+\theta$ | $-\lambda+2\mu$ | $-3\lambda+\mu$ |
| 24 | $\sqrt{-6}$ | $\sqrt{-3}$ | $\theta$ | $-3$ | $2\mu$ | $-3\lambda$ |
| 40 | $\sqrt{-10}$ | $\sqrt{-5}$ | $\theta$ | $-5$ | $2\mu$ | $-5\lambda$ |
| 52 | $\sqrt{-13}$ | $\frac{1+\sqrt{-13}}{\sqrt{2}}$ | $1+\theta$ | $-6+\theta$ | $-\lambda+2\mu$ | $-7\lambda+\mu$ |
| 88 | $\sqrt{-22}$ | $\sqrt{-11}$ | $\theta$ | $-11$ | $2\mu$ | $-11\lambda$ |
| 148 | $\sqrt{-37}$ | $\frac{1+\sqrt{-37}}{\sqrt{2}}$ | $1+\theta$ | $-18+\theta$ | $-\lambda+2\mu$ | $-19\lambda+\mu$ |
| 232 | $\sqrt{-58}$ | $\sqrt{-29}$ | $\theta$ | $-29$ | $2\mu$ | $-29\lambda$ |

TABLE II.

Systems of numbers of the second type; $D=-4\delta+1$, $\theta=\frac{1+\sqrt{D}}{2}$, $\lambda$ and $\mu$ are conjugate.

| $-D$ | $N(\theta)=\delta$ | $\lambda$ | $\lambda^2$ | $\lambda\mu$ | $\lambda\theta$ | $\mu\theta$ |
|---|---|---|---|---|---|---|
| 15 | 4 | $\frac{\sqrt{5}+\sqrt{-3}}{2}$ | $\theta$ | 2 | $\lambda-2\mu$ | $2\lambda$ |
| 35 | 9 | $\frac{\sqrt{7}+\sqrt{-5}}{2}$ | $\theta$ | 3 | $\lambda-3\mu$ | $3\lambda$ |
| 51 | 13 | $\frac{\sqrt{17}+\sqrt{-3}}{2}$ | $3+\theta$ | 5 | $4\lambda-5\mu$ | $5\lambda-3\mu$ |
| 91 | 23 | $\frac{\sqrt{13}+\sqrt{-7}}{2}$ | $1+\theta$ | 5 | $2\lambda-5\mu$ | $5\lambda-\mu$ |
| 115 | 29 | $\frac{\sqrt{23}+\sqrt{-5}}{2}$ | $4+\theta$ | 7 | $5\lambda-7\mu$ | $7\lambda-4\mu$ |
| 123 | 31 | $\frac{\sqrt{41}+\sqrt{-3}}{2}$ | $9+\theta$ | 11 | $10\lambda-11\mu$ | $11\lambda-9\mu$ |
| 187 | 47 | $\frac{\sqrt{17}+\sqrt{-11}}{2}$ | $1+\theta$ | 7 | $2\lambda-7\mu$ | $7\lambda-\mu$ |
| 235 | 59 | $\frac{\sqrt{47}+\sqrt{-5}}{2}$ | $10+\theta$ | 13 | $11\lambda-13\mu$ | $13\lambda-10\mu$ |
| 267 | 67 | $\frac{\sqrt{89}+\sqrt{-3}}{2}$ | $21+\theta$ | 23 | $22\lambda-23\mu$ | $23\lambda-21\mu$ |
| 403 | 101 | $\frac{\sqrt{31}+\sqrt{-13}}{2}$ | $4+\theta$ | 11 | $5\lambda-11\mu$ | $11\lambda-4\mu$ |
| 427 | 107 | $\frac{\sqrt{61}+\sqrt{-7}}{2}$ | $13+\theta$ | 17 | $14\lambda-17\mu$ | $17\lambda-13\mu$ |

TABLE III.

33. The system $D=-20$; $\theta=\sqrt{-5}$.

Table of numbers, their prime factors and norms, in the order of magnitude of their norms.

| Principal Numbers. | | | Secondary Numbers. | | |
|---|---|---|---|---|---|
| Numbers. | Prime Factors. | Norms. | Numbers. | Prime Factors. | Norms. |
| $1$ | – | 1 | $\lambda$ | – | 2 |
| $2$ | $\lambda^2$ | 4 | $\mu$ | – | 3 |
| $\theta$ | – | 5 | $\lambda-\mu$ | – | " |
| $1+\theta$ | $\lambda\mu$ | 6 | $\lambda+\mu$ | – | 7 |
| $1-\theta$ | $\lambda(\lambda-\mu)$ | " | $2\lambda-\mu$ | – | " |
| $3$ | $\mu(\lambda-\mu)$ | 9 | $2\lambda$ | $\lambda^2$ | 8 |
| $2+\theta$ | $-(\lambda-\mu)^2$ | " | $\lambda-2\mu$ | $-\lambda\theta$ | 10 |
| $2-\theta$ | $-\mu^2$ | " | $2\mu$ | $\lambda^2\mu$ | 12 |
| $3+\theta$ | $\lambda(\lambda+\mu)$ | 14 | $2\lambda-2\mu$ | $\lambda^2(\lambda-\mu)$ | " |
| $3-\theta$ | $\lambda(2\lambda-\mu)$ | " | $2\lambda+\mu$ | $\theta(\lambda-\mu)$ | 15 |
| $4$ | $\lambda^4$ | 16 | $3\lambda-\mu$ | $-\theta\mu$ | " |
| $2\theta$ | $\lambda^2\theta$ | 20 | $3\lambda$ | $\lambda\mu(\lambda-\mu)$ | 18 |
| $4+\theta$ | $\mu(2\lambda-\mu)$ | 21 | $\lambda+2\mu$ | $-\lambda(\lambda-\mu)^2$ | " |
| $4-\theta$ | $(\lambda-\mu)(\lambda+\mu)$ | " | $3\lambda-2\mu$ | $-\lambda\mu^2$ | " |
| $1+2\theta$ | $-(\lambda-\mu)(2\lambda-\mu)$ | " | $\lambda-3\mu$ | – | 23 |
| $1-2\theta$ | $-\mu(\lambda+\mu)$ | " | $2\lambda-3\mu$ | – | " |
| $2+2\theta$ | $\lambda^2\mu$ | 24 | $3\mu$ | $\mu^2(\lambda-\mu)$ | 27 |
| $2-2\theta$ | $\lambda^2(\lambda-\mu)$ | " | $3\lambda-3\mu$ | $\mu(\lambda-\mu)^2$ | " |
| $5$ | $-\theta^2$ | 25 | $3\lambda+\mu$ | $-\mu^2$ | " |
| | | | $4\lambda-\mu$ | $-(\lambda-\mu)^2$ | " |

# V. *Partitions of Numbers whose Graphs possess Symmetry.*

By Major P. A. MacMahon, R.A., D.Sc., F.R.S., Hon. Mem. C.P.S.

[*Received* and *read* 28 November, 1898.]

It will be remembered that in *Phil. Trans. R. S. of London*, Vol. 187, 1896 A. pp. 619—673, I undertook the extension to three dimensions of Sylvester's constructive theory of Partitions. In Sylvester's two-dimensional theory every partition of a unipartite number can be associated with a regular two-dimensional graph. In the present theory only a limited number of the partitions of multipartite numbers can be represented by regular graphs in three dimensions. But whereas Sylvester was only concerned with unipartite numbers, the three-dimensional theory has to do with multipartite numbers of unrestricted multiplicity. Though the partitions of such are not all involved the field is infinitely greater, and all which come within the purview of the regular graph are brought harmoniously together. If in this new theory we restrict ourselves to two dimensions but view the graphs from a three-dimensional standpoint, we obtain in general six interpretations of the graphs instead of two and multipartite numbers are brought under consideration as well as those which are unipartite. The enumeration of the three-dimensional graphs of given weight (number of nodes), the numbers of nodes along the axes being restricted not to exceed $l$, $m$, $n$ respectively, was conjectured in Part I. but only established for some particular values of $l$, $m$, $n$.

For $m \geqq l$ it may be written

$$\frac{1-x^{n+1}}{1-x} \cdot \left(\frac{1-x^{n+2}}{1-x^2}\right)^2 \cdots \left(\frac{1-x^{n+l-1}}{1-x^{l-1}}\right)^{l-1}$$

$$\times \left\{\frac{1-x^{n+l}}{1-x^l} \cdot \frac{1-x^{n+l+1}}{1-x^{l+1}} \cdots \frac{1-x^{n+m}}{1-x^m}\right\}^l$$

$$\times \left(\frac{1-x^{n+m+1}}{1-x^{m+1}}\right)^{l-1} \cdot \left(\frac{1-x^{n+m+2}}{1-x^{m+2}}\right)^{l-2} \cdots \frac{1-x^{n+l+m-1}}{1-x^{l+m-1}}.$$

The symmetry of this expression and its real nature are best shewn by a symbolic crystalline form.

Observing that it is composed of factors of the form $(1-x^s)^t$, where $t$ may be positive or negative, put

$$1-x^s = \exp.(-u^s) \text{ in the case of every factor,}$$

and it will be found, after a few simplifications, to take the form

$$\exp.\frac{u}{(1-u)^2}(1-u^l)(1-u^m)(1-u^n).$$

In the two-dimensional theory the generating function

$$\frac{(1-x^{l+1})(1-x^{l+2})\ldots(1-x^{l+m})}{(1-x)(1-x^2)\ldots(1-x^m)}$$

has the symbolic crystalline form

$$\exp.\frac{u}{1-u}(1-u^l)(1-u^m),$$

whilst in one-dimensional theory

$$\frac{1-x^{l+1}}{1-x}$$

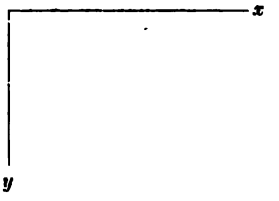
obviously leads to

$$\exp.\, u(1-u^l).$$

Hence we seem to have before us a system in $\kappa$ dimensions associated with the crystalline form

$$\exp.\frac{u}{(1-u)^{\kappa-1}}(1-u^{l_1})(1-u^{l_2})\ldots(1-u^{l_\kappa}).$$

In general a graph by rotations about the axes of $x$, $y$, and $z$




may assume six forms.

When these forms are identical the graph is said to be symmetrical or to possess $xyz$-symmetry.

Such ex. gr. is

When the six forms reduce to three the graph is said to be quasi-symmetrical. If it be such that each layer of nodes is symmetrical in two dimensions or, the same thing, is a Sylvester self-conjugate graph, it is said to possess $xy$-symmetry. Ex. gr.

Similarly the graph

$$\bullet\ \bullet\ \bullet\ \bullet$$

possesses $yz$-symmetry, and by rotation about the $y$ axis, or that of $z$, may be converted into one possessing $xy$ or $zx$-symmetry.

It is proposed to investigate generating functions for the enumeration of graphs possessing $xy$ and $xyz$-symmetry, the former naturally including the latter.

Algebraic theorems will be evolved in the course of the work by the method initiated by Sylvester.

### $xy$-SYMMETRICAL GRAPHS.

The self-conjugate Sylvester graphs which have $i$ nodes along each axis can be formed by fitting into an angle of $2i-1$ nodes any number of angles of nodes, any angle containing an uneven number, less than $2i-1$, of nodes and no two angles possessing the same number of nodes. Ex. gr. for $i=7$ we have the angles

$$\begin{array}{cccc} \bullet\ \bullet\ \bullet\ \bullet & \bullet\ \bullet\ \bullet & \bullet\ \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

which by selection of the 1st, 3rd and 4th of the angles may be formed up into the graph

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array}$$

Hence, as Sylvester shewed, the generating function of such graphs is immediately seen to be

$$x^{2i-1}(1+x)(1+x^3)\dots(1+x^{2i-3}).$$

Each layer of the three-dimensional graph has this form, and if there be two layers at most we may construct a generating function

$$\Omega a_1 a_2 \dots a_i x^{2i-1}(1+a_1x)(1+a_1a_2x^3)\dots(1+a_1a_2\dots a_{i-1}x^{2i-3})$$
$$\times\left\{\left(1+\frac{x}{a_1}\right)\left(1+\frac{x^3}{a_1a_2}\right)\left(1+\frac{x^5}{a_1a_2a_3}\right)\dots \text{ ad inf.}\right\}$$

where $\Omega$ is a symbol of Cayley's which means that after multiplication all terms containing negative powers of $a_1$, $a_2$, $a_3 \dots a_i$ are to be struck out and then each of these letters put equal to unity.

The first line of the expression following $\Omega$ is derived from

$$x^{2i-1}(1+x)(1+x^3)\dots(1+x^{2i-3})$$

by placing as coefficient to each $x^{2s-1}$ the product $a_1a_2\dots a_s$.

The angles of the first or lower layer correspond to the powers of $x$ in the first line, those of the second layer to the powers in the second line, and the operation of $\Omega$ is such as to prevent any combinations of the former and the latter which give rise to an irregular graph.

Summing this function from $i=1$ to $i=i$ and supposing its value unity when $i=0$ (a convention that is made only for convenience; no form exists for $i=0$) we obtain

$$\Omega(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\dots(1+a_1a_2\dots a_ix^{2i-1})$$
$$\times\left\{\left(1+\frac{x}{a_1}\right)\left(1+\frac{x^3}{a_1a_2}\right)\left(1+\frac{x^5}{a_1a_2a_3}\right)\dots\text{ ad inf.}\right\}$$

as the generating function which enumerates $xy$-symmetrical graphs of at most two layers, the number of nodes along an $x$ or $y$ axis being limited not to exceed $i$.

Further if $i$ be infinite this becomes:—

$$\Omega(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\dots\dots\text{ ad inf.}$$
$$\times\left(1+\frac{x}{a_1}\right)\left(1+\frac{x^3}{a_1a_2}\right)\left(1+\frac{x^5}{a_1a_2a_3}\right)\dots\dots\text{ ad inf.}$$

It is moreover clear that the generating function of $xy$-symmetrical graphs which have at most $i$ nodes along each of the axes $x$, $y$ and at most $j$ nodes along the axis of $z$ (i.e. which involve at most $j$ layers) is:—

$$\Omega(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\dots\dots(1+a_1a_2\dots a_ix^{2i-1})$$
$$\times\left(1+\frac{b_1}{a_1}x\right)\left(1+\frac{b_1b_2}{a_1a_2}x^3\right)\left(1+\frac{b_1b_2b_3}{a_1a_2a_3}x^5\right)\dots\dots\text{ ad inf.}$$
$$\times\left(1+\frac{c_1}{b_1}x\right)\left(1+\frac{c_1c_2}{b_1b_2}x^3\right)\left(1+\frac{c_1c_2c_3}{b_1b_2b_3}x^5\right)\dots\dots\text{ ad inf.}$$
$$\times\left(1+\frac{d_1}{c_1}x\right)\left(1+\frac{d_1d_2}{c_1c_2}x^3\right)\left(1+\frac{d_1d_2d_3}{c_1c_2c_3}x^5\right)\dots\dots\text{ ad inf.}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$j$ rows,

$\Omega$ operating in regard to all the symbols, $a$, $b$, $c$, $d$, &c. ...

If the graphs be unrestricted, as regards $i$, we put $i=\infty$; and, if they be totally unrestricted, we regard the tableau, upon which $\Omega$ operates, as possessing an unlimited number of rows and columns.

The generating function is crude. One, which only involves $x$, is ultimately to be desired. It should be possible, by algebraic processes, to perform the operation $\Omega$ and thus to pick out the terms of the product which constitute the reduced generating function. This appears to be a matter of considerable difficulty, and in order to determine the probable form of the reduced function I have examined many particular cases and

attempted its construction. My conclusion is that, writing $(s)$ to denote $1-x^s$, the reduced function is, in all probability, an algebraic fraction of which the numerator is

$$\begin{array}{ll} (j+1)(j+3)(j+5) & \ldots\ldots(j+2i-1) \\ \times(2j+4)(2j+6)(2j+8) & \ldots\ldots(2j+4i-4) \\ \times(2j+8)(2j+10)(2j+12) & \ldots\ldots(2j+4i-8) \\ \times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \times(2j+4s)(2j+4s+2)(2j+4s+4)\ldots\ldots(2j+4i-4s) & \\ \times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{array}$$

wherein, if $i$ be even, there are $\frac{1}{2}i$ rows the last of which is

$$(2j+2i);$$

and, if $i$ be uneven, there are $\frac{1}{2}(i-1)$ rows the last of which is

$$(2j+2i-2)(2j+2i)(2j+2i+2);$$

and the denominator is obtained from the numerator by putting $j=0$, viz.:—it is

$$\begin{array}{ll} (1)(3)(5) & \ldots\ldots(2i-1) \\ \times(4)(6)(8) & \ldots\ldots(4i-4) \\ \times(8)(10)(12) & \ldots\ldots(4i-8) \\ \times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \times(4s)(4s+2)(4s+4)\ldots\ldots(4i-4s) & \\ \times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{array}$$

the last row being $(2i)$ or $(2i-2)(2i)(2i+2)$ according as $i$ is even or uneven.

The proof of this formula, the truth of which seems unquestionable, is much to be desired.

When the number of layers of nodes is unrestricted we put $j=\infty$ and the numerator reduces to unity. When moreover both $i$ and $j$ are unrestricted in magnitude the reduced function becomes

$$\frac{1}{(1)(3)(5)(7)\ldots(4)(6)(8)^2(10)^2(12)^3(14)^3(16)^4(18)^4\ldots\ldots},$$

or as it may be also written

$$\frac{(1+x)(1+x^3)(1+x^5)(1+x^7)\ldots\ldots}{(2)(4)(6)^2(8)^2(10)^3(12)^3(14)^4(16)^4\ldots\ldots},$$

wherein the numerator denotes the generating function of Sylvester's unrestricted self-conjugate graphs in two dimensions.

Some particular cases are interesting.

By putting $j=1$ we should obtain Sylvester's result in two dimensions.

We find

$$\frac{(2)\,(6)\,(10)\,(14)\ldots(4i-2)}{(1)\,(3)\,(5)\,(7)\ldots(2i-1)},$$

which may be written

$$(1+x)(1+x^3)(1+x^5)(1+x^7)\ldots(1+x^{2i-1})$$

and is right.

When $j=2$, we find

$$\frac{(2i+1)}{(1)}\cdot\frac{(2i+4)\,(2i+6)\ldots(4i-2)\,(4i)}{(4)\quad(6)\quad\ldots(2i-2)\,(2i)},$$

or

$$\frac{(2i+2)\,(2i+4)\ldots(4i-2)\,(4i)}{(2)\quad(4)\quad\ldots(2i-2)\,(2i)}+x\,\frac{(2i+4)\,(2i+6)\ldots(4i-2)\,(4i)}{(2)\;(4)\;(6)\quad\ldots(2i-2)}.$$

For an even weight $2w$ we must take the coefficients of $x^w$ in

$$\frac{(i+1)\,(i+2)\ldots(2i-1)\,(2i)}{(1)\quad(2)\quad\ldots\;(i-1)\;(i)},$$

and this is the generating function of two-dimensional graphs of weight $w$, not more than $i$ nodes being allowed along either the $x$ or $y$ axis. Hence a correspondence between the at-most-two-layer $xy$-symmetrical graphs of weight $2w$ restricted as to the $x$ and $y$ axes by the number $i$ and the graphs in two dimensions of weight $w$ restricted as to the axes by the number $i$.

Ex. gr. for $w=4$, $i=3$ the correspondence is

| | | | | | |
|---|---|---|---|---|---|
| 1 1 1 | 1 1 | 1 1 | 2 2 1 | 2 2 | 1 1 1 |
| 1 | 1 1 | 1 | 2 | 2 2 | 1 1 1 |
| | | 1 | 1 | | 1 1 |

For an uneven weight $2w+1$ we take the coefficients of $x^w$ in

$$\frac{(i+2)\,(i+3)\ldots(2i-1)\,(2i)}{(1)\,(2)\,(3)\;\ldots\;(i-1)},$$

and this is the generating function of two-dimensional graphs of weight $w$, not more than $i+1$ nodes being allowed along the $x$ axis nor more than $i-1$ along the $y$ axis.

The correspondence established is that between the at-most-two-layer $xy$-symmetrical graphs of weight $2w+1$ restricted as to each of the $x$ and $y$ axes by the number $i$ and the graphs of two dimensions of weight $w$ restricted as to the $x$ axis by the number $i+1$ and as to the $y$ axis by the number $i-1$.

Ex. gr. for $w=5$, $i=4$ we have the five to five correspondence

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 1 1 1 1 | 1 1 1 1 | 1 1 1 | 1 1 1 | 1 1 | 1 1 1 1 | 2 1 1 1 | 2 2 1 1 | 2 2 1 | 2 2 2 |
| | 1 | 1 1 | 1 | 1 1 | 1 1 1 | 1 1 1 | 2 1 | 2 1 1 | 2 1 |
| | | | 1 | 1 | 1 1 1 | 1 1 | 1 | 1 1 | 2 |
| | | | | | 1 | 1 | 1 | | |

where $i$ is indefinite the generating function becomes

$$\frac{1+x}{(1-x^2)(1-x^4)(1-x^6)\ldots\ldots \text{ ad inf.}}.$$

This curious result shews that the number of at-most-two-layer $xy$-symmetrical graphs of weight $w$ is equal to the whole number of partitions of $\frac{1}{2}w$ or of $\frac{1}{2}(w-1)$ according as $w$ is even or uneven.

There is another solution of the problem that has been under consideration.

Instead of constructing a generating function from successive layers of nodes parallel to the plane of $xy$, we may build one up by first considering all the exterior angles of nodes; then those which become exterior when the former are removed; and so on. Thus if any graph were

```
4 3 2 2 1
3 2 2 1
2 2 1 1
2 1 1
1
```

we first take

```
4 3 2 2 1
3
2
2
1
```

as constructed by the superposition of

```
1 1 1 1 1   1 1 1 1   1 1   1;
1           1         1
1           1
1           1
1
```

then

```
2 2 1   made up of   1 1 1   1 1
2                    1       1     ;
1                    1
```

then

```
1 1
1 .
```

We are then led to the crude generating function

$$\Omega\frac{1}{(1-m)(1-mx)(1-max^3)(1-mabx^5)(1-mabcx^7)\ldots(1-mabc\ldots x^{2i-1})}$$

$$\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\ldots \text{ ad inf.},$$

$$\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\ldots \text{ ad inf.}$$

$$\vdots$$

ad inf.

in the ascending expansion of which we must take the coefficient of $m^j$, $\Omega$ operating in regard to the letters

$$\begin{array}{l} a\,,\ b\,,\ c\,,\ldots \\ a'\,,\ b'\,,\ c'\,,\ldots \\ a''\,,\ b''\,,\ c''\,,\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

We have, therefore, the identity

$$\begin{aligned} &\Omega\,(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\ldots(1+a_1a_2\ldots a_ix^{2i-1}) \\ &\qquad\times\left(1+\frac{b_1}{a_1}x\right)\left(1+\frac{b_1b_2}{a_1a_2}x^3\right)\left(1+\frac{b_1b_2b_3}{a_1a_2a_3}x^5\right)\ldots \text{ ad inf.} \\ &\qquad\times\left(1+\frac{c_1}{b_1}x\right)\left(1+\frac{c_1c_2}{b_1b_2}x^3\right)\left(1+\frac{c_1c_2c_3}{b_1b_2b_3}x^5\right)\ldots \text{ ad inf.} \\ &\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\qquad\qquad j \text{ rows} \\ &= \mathrm{Co}\, m^j\,\Omega\,\frac{1}{(1-m)(1-mx)(1-max^3)(1-mabx^5)(1-mabcx^7)\ldots(1-mabc\ldots x^{2i-1})} \\ &\qquad\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\ldots \text{ ad inf.} \\ &\qquad\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\ldots \text{ ad inf.} \\ &\qquad\qquad\vdots \\ &\qquad\qquad\text{ad inf.} \end{aligned}$$

and, when $j$ is unrestricted,

$$\begin{aligned} &\Omega\,(1+a_1x)(1+a_1a_2x^3)(1+a_1a_2a_3x^5)\ldots(1+a_1a_2\ldots a_ix^{2i-1}) \\ &\qquad\times\left(1+\frac{b_1}{a_1}x\right)\left(1+\frac{b_1b_2}{a_1a_2}x^3\right)\left(1+\frac{b_1b_2b_3}{a_1a_2a_3}x^5\right)\ldots \text{ ad inf.} \\ &\qquad\times\left(1+\frac{c_1}{b_1}x\right)\left(1+\frac{c_1c_2}{b_1b_2}x^3\right)\left(1+\frac{c_1c_2c_3}{b_1b_2b_3}x^5\right)\ldots \text{ ad inf.} \\ &\qquad\qquad\vdots \\ &\qquad\qquad\text{ad inf.} \\ &= \Omega\,\frac{1}{(1-x)(1-ax^3)(1-abx^5)(1-abcx^7)\ldots(1-abc\ldots x^{2i-1})} \\ &\qquad\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\ldots \text{ ad inf.} \\ &\qquad\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\ldots \text{ ad inf.} \\ &\qquad\qquad\vdots \\ &\qquad\qquad\text{ad inf.,} \end{aligned}$$

a remarkable result, which it would be difficult to establish algebraically.

As it is necessary in the sequel we will now determine the generating function which enumerates the $xy$-symmetrical graphs, limited as above, but subject to a new restriction, viz. each layer of nodes is to be formed by, at most, $s$ plane angles of nodes.

The enumeration, it is easy to see, is given by the coefficients of $m^s x^w$ in the development of

$$\frac{1}{1-m}\,\Omega\,(1+ma_1x)(1+ma_1a_2x^3)(1+ma_1a_2a_3x^5)\ldots(1+ma_1a_2\ldots a_ix^{2i-1})$$
$$\times\left(1+\frac{b_1}{a_1}x\right)\left(1+\frac{b_1b_2}{a_1a_2}x^3\right)\left(1+\frac{b_1b_2b_3}{a_1a_2a_3}x^5\right)\ldots \text{ ad inf.}$$
$$\times\left(1+\frac{c_1}{b_1}x\right)\left(1+\frac{c_1c_2}{b_1b_2}x^3\right)\left(1+\frac{c_1c_2c_3}{b_1b_2b_3}x^5\right)\ldots \text{ ad inf.}$$
$$\vdots$$
$$j \text{ rows,}$$

and also by the coefficients of $m^j x^w$ in the development of

$$\frac{1}{1-m}\,\Omega\,\frac{1}{(1-mx)(1-max^3)(1-mabx^5)(1-mabcx^7)\ldots(1-mabc\ldots x^{2i-1})}$$
$$\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\ldots \text{ ad inf.}$$
$$\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\ldots \text{ ad inf.}$$
$$\vdots$$
$$s \text{ rows.}$$

Let the coefficients of $m^s$ in the former of these generating functions be denoted by $F_{j,s}(x)$, and denoting the generating functions by $A$ and $B$ respectively, we have:—

$$A = 1 + mF_{j,1}(x) + m^2F_{j,2}(x) + \ldots + m^sF_{j,s}(x) + \ldots,$$
$$B = 1 + mF_{1,s}(x) + m^2F_{2,s}(x) + \ldots + m^jF_{j,s}(x) + \ldots.$$

Moreover for $j=\infty$, we have

$$A = 1 + mF_{\infty,1}(x) + m^2F_{\infty,2}(x) + m^3F_{\infty,3}(x) + \ldots,$$

and for $s=\infty$,

$$B = 1 + mF_{1,\infty}(x) + m^2F_{2,\infty}(x) + m^3F_{3,\infty}(x) + \ldots.$$

## THE $xyz$-SYMMETRICAL GRAPHS.

Just as Sylvester dissected the $xy$-symmetrical graph in two dimensions into plane angles we may dissect the $xyz$-symmetrical graphs in three dimensions into solid angles. Each solid angle is in the shape of a symmetrical fragment of half of a hollow cube.

In each of the planes $xy$, $yz$, $zx$ we find the same symmetrical two-dimensional graph. If this graph has $i$ columns or rows the number of nodes which lie on one or other of the three axes is $1+3(i-1)$ or $3i-2$. In the plane of $xy$ we can place plane angles of

nodes so as to form a symmetrical graph in two dimensions. If $w$ be the weight of the solid angle we have $w-3i+2$ nodes to dispose symmetrically in the three planes and this can be done in a number of ways which is given by the coefficients of

$$x^{\frac{1}{2}(w-3i+2)}$$

in

$$(1+x)(1+x^3)(1+x^5)\dots(1+x^{2i-3}),$$

that is, by the coefficients of $x^w$ in

$$Q_i = x^{3i-2}(1+x^3)(1+x^9)(1+x^{15})\dots(1+x^{6i-9}),$$

which is therefore the generating function of the solid angles in question which have exactly $i$ nodes along each axis. Observe that $i-1$ factors follow $x^{3i-2}$, and that, when it is convenient, we suppose the expression to have the value unity when $i=0$.

Hence the solid angles which possess $i$ or fewer nodes along the axes are enumerated by

$$\begin{aligned} &1 \\ &+x \\ &+x^4(1+x^3) \\ &+x^7(1+x^3)(1+x^9) \\ &+\dots\dots \\ &+x^{3i-2}(1+x^3)(1+x^9)(1+x^{15})\dots(1+x^{6i-9}). \end{aligned}$$

Fitting solid angle graphs together when possible produces $xyz$-symmetrical graphs.

When $i=2$, $Q_2 = x^4(1+x^3)$, the two solid angles being

$$\begin{matrix} \odot & \bullet \\ \bullet & \end{matrix} \qquad \begin{matrix} \odot & \odot \\ \odot & \bullet \end{matrix}$$

of contents 4 and 7 respectively.

We cannot fit a solid angle into the first of these, for there is no node upon which it can rest. In the case of the second we can fit in the solid angle for which $i=1$, $Q_1 = x$ represented by a single node $\bullet$, and thus form the symmetrical graph

$$\begin{matrix} \odot & \odot \\ \odot & \odot \end{matrix} \quad \text{of content 8.}$$

Synthetically we form the generating function

$$\Omega x^4\left(1+ax^3\right)\left(1+\frac{1}{a}x\right) = x^4+x^7+x^8$$

of all symmetric graphs having $i=2$.

Observe that the construction of the factors, following the operator $\Omega$, permits the association of

$$\begin{matrix} \odot & \odot \\ \odot & \bullet \end{matrix} \quad \text{and} \quad \bullet$$

and does not permit that of

$$\begin{matrix} \odot & \bullet \\ \bullet & \end{matrix} \quad \text{and} \quad \bullet$$

Restricting ourselves to two solid angles when

$$i = 3, \quad Q_3 = x^7 (1 + x^3)(1 + x^9)$$

we are similarly led to the construction of the generating function

$$\Omega x^7 (1 + ax^3)(1 + abx^9) \times \left\{1 + \frac{x}{a} + \frac{x^4}{ab}\left(1 + \frac{x^3}{a}\right)\right\},$$

whence after expansion and operation we find

$$(x^7 + x^{10} + x^{16} + x^{19}) + (x^{11} + x^{17} + x^{20}) + (x^{20} + x^{28}) + x^{30},$$

and the correspondence is

| $x^7$ | $x^7 . ax^3$ | $x^7 \times abx^9$ | $x^7 . ax^3 . abx^9$ |
|---|---|---|---|
| 3 1 1 | 3 2 1 | 3 3 2 | 3 3 3 |
| 1 | 2 1 | 3 1 1 | 3 1 1 |
| 1 | 1 | 2 1 | 3 1 1 |

| $x^7 . ax^3 . \frac{x}{a}$ | $x^7 . abx^9 . \frac{x}{a}$ | $x^7 . ax^3 . abx^9 . \frac{x}{a}$ |
|---|---|---|
| 3 2 1 | 3 3 2 | 3 3 3 |
| 2 2 | 3 2 1 | 3 2 1 |
| 1 | 2 1 | 3 1 1 |

| $x^7 . abx^9 . \frac{x^4}{ab}$ | $x^7 . ax^3 . abx^9 . \frac{x^4}{ab}$ |
|---|---|
| 3 3 2 | 3 3 3 |
| 3 3 2 | 3 3 2 |
| 2 2 | 3 2 1 |

$$x^7 . ax^3 . abx^9 . \frac{x^4}{ab} . \frac{x^3}{a} .$$

3 3 3
3 3 3
3 3 2

In the form which arises from the product $x^7 . abx^9 . \frac{x}{a}$, the largest solid angle is given by $x^7 . abx^9$; that is, $x^7$ gives the axial portion $\begin{matrix} 3\ 1\ 1 \\ 1 \\ 1 \end{matrix}$, $x^9$ yields $\begin{matrix} 1\ 1 \\ 1 \end{matrix}$ in each of the three planes, so that the resulting angle is $\begin{matrix} 3\ 3\ 2 \\ 3\ 1\ 1 \\ 2\ 1 \end{matrix}$; the next largest solid angle is given by $\frac{x}{a}$ and this fits into the larger.

Again from $x^7 . ax^3 . abx^9 . \frac{x^4}{ab} . \frac{x^3}{a}$ we get first $\begin{matrix} & 3 & 1 & 1 \\ & 1 & & \\ & 1 & & \end{matrix}$ from $x^7$, and then $\begin{matrix} 1 & 1 \\ 1 & \end{matrix}$ and 1, in each plane, from $x^9$ and $x^3$, yielding $\begin{matrix} 3 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{matrix}$ the outer solid angle; and $\frac{x^4}{ab} . \frac{x^3}{a}$ gives a solid angle, composed of $\begin{matrix} 2 & 1 \\ 1 & \end{matrix}$ and 1 fitting in each plane, viz.:— $\begin{matrix} 2 & 2 \\ 2 & 1 \end{matrix}$, and this fits into the larger solid angle yielding $\begin{matrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 2 \end{matrix}$.

It will be clear now that the generating function for symmetrical graphs having $i$ nodes along each axis and formed of at most two solid angles is

$$\Omega x^{3i-2}(1 + a_1x^3)(1 + a_1a_2x^9) \dots (1 + a_1a_2 \dots a_{i-1}x^{6i-9})$$

$$\times \left\{1 + \frac{x}{a_1} + \frac{x^4}{a_1a_2}\left(1 + \frac{x^3}{a_1}\right) + \frac{x^7}{a_1a_2a_3}\left(1 + \frac{x^3}{a_1}\right)\left(1 + \frac{x^9}{a_1a_2}\right) + \dots \text{ ad inf.}\right\},$$

the general term in the series to infinity being

$$\frac{x^{3s-5}}{a_1a_2 \dots a_{s-1}}\left(1 + \frac{x^3}{a_1}\right)\left(1 + \frac{x^9}{a_1a_2}\right) \dots \left(1 + \frac{x^{6s-15}}{a_1a_2 \dots a_{s-2}}\right).$$

Summing this function, for values of $i$, it is found that the generating function, for the graphs composed of at most two solid angles and having at most $i$ nodes along each axis, is

$$\Omega \{1 + x + x^4(1 + a_1x^3) + x^7(1 + a_1x^3)(1 + a_1a_2x^9) + \dots$$
$$+ x^{3i-2}(1 + a_1x^3)(1 + a_1a_2x^9) \dots (1 + a_1a_2 \dots a_{i-1}x^{6i-9})\}$$

$$\times \left\{1 + \frac{x}{a_1} + \frac{x^4}{a_1a_2}\left(1 + \frac{x^3}{a_1}\right) + \frac{x^7}{a_1a_2a_3}\left(1 + \frac{x^3}{a_1}\right)\left(1 + \frac{x^9}{a_1a_2}\right) + \dots \text{ ad inf.}\right\}.$$

If the graphs are to be composed of at most two solid angles but to be otherwise unrestricted we obtain

$$\Omega \{1 + x + x^4(1 + a_1x^3) + x^7(1 + a_1x^3)(1 + a_1a_2x^9) + \dots \text{ ad inf.}\}$$

$$\times \left\{1 + \frac{x}{a_1} + \frac{x^4}{a_1a_2}\left(1 + \frac{x^3}{a_1}\right) + \frac{x^7}{a_1a_2a_3}\left(1 + \frac{x^3}{a_1}\right)\left(1 + \frac{x^9}{a_1a_2}\right) + \dots \text{ ad inf.}\right\}.$$

It is now easy to pass to the general case in which the composition is to be from at most $s$ solid angles. The generating function is

$$\Omega\,\{1 + x + x^4(1 + a_1x^3) + x^7(1 + a_1x^3)(1 + a_1a_2x^9) + \dots$$
$$+ x^{3i-2}(1 + a_1x^3)(1 + a_1a_2x^9)\dots(1 + a_1a_2\dots a_{i-1}x^{3i-9})\}$$
$$\times\left\{1 + \frac{x}{a_1} + \frac{x^4}{a_1a_2}\left(1 + \frac{b_1}{a_1}x^3\right) + \frac{x^7}{a_1a_2a_3}\left(1 + \frac{b_1}{a_1}x^3\right)\left(1 + \frac{b_1b_2}{a_1a_2}x^9\right) + \dots \text{ ad inf.}\right\}$$
$$\times\left\{1 + \frac{x}{b_1} + \frac{x^4}{b_1b_2}\left(1 + \frac{c_1}{b_1}x^3\right) + \frac{x^7}{b_1b_2b_3}\left(1 + \frac{c_1}{b_1}x^3\right)\left(1 + \frac{c_1c_2}{b_1b_2}x^9\right) + \dots \text{ ad inf.}\right\}$$
$$\times\left\{1 + \frac{x}{c_1} + \frac{x^4}{c_1c_2}\left(1 + \frac{d_1}{c_1}x^3\right) + \frac{x^7}{c_1c_2c_3}\left(1 + \frac{d_1}{c_1}x^3\right)\left(1 + \frac{d_1d_2}{c_1c_2}x^9\right) + \dots \text{ ad inf.}\right\}$$
$$\times \dots\dots$$

$s$ rows.

When the first row is also continued to infinity, and the number of rows is infinite, we have the crude form of generating function for $xyz$-symmetrical graphs quite unrestricted.

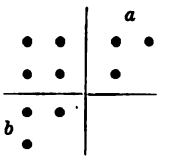
When $s = 1$ and $i = \infty$ it may be easily proved that the generating function may be written

$$1 + \frac{x}{1 - x^3} + \frac{x^7}{(1 - x^3)(1 - x^9)} + \frac{x^{19}}{(1 - x^3)(1 - x^9)(1 - x^{15})} + \dots$$
$$+ \frac{x^{(k-1)^2 - (k-2)^2}}{(1 - x^3)(1 - x^9)(1 - x^{15})\dots(1 - x^{6k-9})} + \dots$$

There is another mode of enumeration of $xyz$-symmetrical graphs which it is important to consider.

Durfee has shewn how to dissect a symmetrical graph in two dimensions into a square of nodes and two appendages lateral and subjacent.

Ex. gr. the graph




where this is a square of four nodes, a lateral appendage $a$ and one which is subjacent $b$. This dissection leads to the expression of the generating function in the form of an infinite series of algebraic fractions. Sylvester further applied the same dissection to unsymmetrical graphs and derived algebraic identities of great interest.

In the case of three dimensions we also have a dissection of the same nature. This is not based upon the isolation of a cube of nodes as might at first appear.

If we take such a cube, for example,

2 2
2 2

we may, it is true, attach appropriate lateral, subjacent, and superjacent graphs and thus obtain an $xyz$-symmetrical graph; but a slight consideration shews that a large number of symmetrical graphs escape enumeration by this process. Ex. gr. the graph

$$\begin{array}{cc|c} 3 & 3 & 2 \\ 3 & 2 & 1 \\ \hline 2 & 1 & \end{array}$$

is based upon the cube in question, whereas the graph

$$\begin{array}{cc|c} 3 & 3 & 3 \\ 3 & 2 & 1 \\ \hline 3 & 1 & 1 \end{array}$$

is not based upon that or any other cube, yet it is without doubt symmetrical.

In the former of the two graphs observe that the appendages are

$$\begin{array}{rl} \text{lateral} & \begin{array}{l} 2 \\ 1 \end{array} \\ \text{subjacent} & 2\ 1 \\ \text{superjacent} & \begin{array}{l} 1\ 1 \\ 1 \end{array} . \end{array}$$

The fact is that symmetrical graphs are based also upon graphs other than those which are perfect cubes.

The whole series is formed as follows:—

We have, first, those based upon the cube 1, viz.

the base is 1.

Secondly, we have those based upon graphs such that there is a square of four nodes in each of the three planes of reference. These are of two kinds, viz.:—

$$\begin{array}{cc} 2\ 2 & 2\ 2 \\ 2\ 1 & 2\ 2 \end{array}$$

where the nodes of the former are in the shape of the half of a hollow cube and the latter is obtained from the former by combining with it the cube 1.

Thirdly, we have four bases derived from the graph which has the shape of a half-hollow-cube of side 3; viz.:—

$$\begin{array}{ccccccc} 3\ 3\ 3 & \quad & 3\ 3\ 3 & \quad & 3\ 3\ 3 & \quad & 3\ 3\ 3 \\ 3\ 1\ 1 & & 3\ 2\ 1 & & 3\ 3\ 3 & & 3\ 3\ 3 \\ 3\ 1\ 1 & & 3\ 1\ 1 & & 3\ 3\ 2 & & 3\ 3\ 3 \end{array}$$

where observe that the three latter bases are derived from the former by combination with the three bases previously constructed, viz.:—

$$1 \quad \begin{array}{cc} 2\ 2 & 2\ 2 \\ 2\ 1 & 2\ 2 \end{array}.$$

Similarly, of the fourth order, we have eight bases, viz.:—

| | | | |
|---|---|---|---|
| 4 4 4 4 | 4 4 4 4 | 4 4 4 4 | 4 4 4 4 |
| 4 1 1 1 | 4 2 1 1 | 4 3 3 1 | 4 3 3 1 |
| 4 1 1 1 | 4 1 1 1 | 4 3 2 1 | 4 3 3 1 |
| 4 1 1 1 | 4 1 1 1 | 4 1 1 1 | 4 1 1 1 |
| 4 4 4 4 | 4 4 4 4 | 4 4 4 4 | 4 4 4 4 |
| 4 4 4 4 | 4 4 4 4 | 4 4 4 4 | 4 4 4 4 |
| 4 4 2 2 | 4 4 3 2 | 4 4 4 4 | 4 4 4 4 |
| 4 4 2 2 | 4 4 2 2 | 4 4 4 3 | 4 4 4 4 |

the seven latter being derived from the former by combination with the seven forms previously constructed.

The way in which the bases are built up is now plain and we see that, of order $n$, we can construct $2^{n-1}$ bases of which $2^{n-1}-1$ are derived by combining with the half-hollow-square of order $n$, all the bases of lower orders in number,

$$1+2+2^2+\ldots+2^{n-2}=2^{n-1}-1.$$

As one illustration take the graph

$$\begin{array}{c|c|c}
O & \multicolumn{2}{c}{x} \\
 & & a \\
 & 6\ 5\ 4 & 3\ 2\ 1 \\
 & c\ 5\ 2\ 1 & 1\ 1 \\
 & 4\ 1\ 1 & 1 \\
\hline
 & 3\ 1\ 1 & \\
 & b\ 2\ 1 & \\
y & 1 &
\end{array}$$

the $z$ axis being perpendicular to the plane of the paper.

The graph is built upon the base

$$\begin{array}{l}
3\ 3\ 3 \\
3\ 2\ 1 \\
3\ 1\ 1;
\end{array}$$

$a$ the lateral appendage being

$$\begin{array}{l}
3\ 2\ 1 \\
1\ 1 \\
1;
\end{array}$$

$b$ the subjacent appendage being

$$\begin{array}{l}
3\ 1\ 1 \\
2\ 1 \\
1;
\end{array}$$

$c$ the superjacent appendage being

$$\begin{array}{l}
3\ 2\ 1 \\
2 \\
1.
\end{array}$$

From the lateral appendage we derive in succession the subjacent and superjacent appendages.

The rule is to face the origin and give the lateral graph right-handed rotations through 90° about the axes of $z$ and $y$ in succession. We thus derive the subjacent graph, and a repetition of the process upon the latter then gives the superjacent graph.

Thus starting with the lateral

$$\begin{array}{l} 3\ 2\ 1 \\ 1\ 1 \\ 1 \end{array},$$

the two rotations give in succession

$$\begin{array}{l} 1\ 1\ 3 \\ \ \ 1\ 2 \\ \ \ \ \ 1 \end{array} \quad \text{and} \quad \begin{array}{l} 3\ 1\ 1 \\ 2\ 1 \\ 1 \end{array},$$

the latter being the subjacent, and operating similarly on the latter we obtain in succession

$$\begin{array}{l} 1\ 2\ 3 \\ \ \ 1\ 1 \\ \ \ \ \ 1 \end{array} \quad \text{and} \quad \begin{array}{l} 3\ 2\ 1 \\ 2 \\ 1 \end{array},$$

the last written graph being the superjacent.

As another example, if the lateral be

$$\begin{array}{l} 2\ 2\ 1 \\ 1\ 1 \end{array},$$

we obtain by operation

$$\begin{array}{l} 1\ 2 \\ 1\ 2 \\ \ \ 1 \end{array} \quad \text{and} \quad \begin{array}{l} 2\ 1 \\ 2\ 1 \\ 1 \end{array},$$

giving $\begin{array}{l} 2\ 1 \\ 2\ 1 \\ 1 \end{array}$ the subjacent; operating upon this

$$\begin{array}{l} 1\ 2\ 2 \\ \ \ 1\ 1 \end{array} \quad \text{and} \quad \begin{array}{l} 3\ 2 \\ 2 \end{array}$$

giving $\begin{array}{l} 3\ 2 \\ 2 \end{array}$ the superjacent.

Compare the graph

$$\begin{array}{l|l} 5\ 4 & 2\ 2\ 1 \\ 4\ 1 & 1\ 1 \\ \hline 2\ 1 & \\ 2\ 1 & \\ 1 & \end{array}$$

upon the base

$$\begin{array}{l} 2\ 2 \\ 2\ 1 \end{array}.$$

We have arrived at the point of shewing the construction of the bases and we have seen how to construct the graph, being given the base and the lateral; the base and the lateral completely determine the graph, and if they be of contents $w_1$, $w_2$ respectively the complete graph is of content $w_1 + 3w_2$. For a given base we have now to determine the possible forms of lateral appendage preparatory to attempting their enumeration.

Every line of numbers parallel to the axis of $y$ in a symmetrical graph is of necessity a self-conjugate partition of a number, for otherwise more than one interpretation of the graph would be obtainable. Ex. gr. in the graph

$$\begin{array}{l} 5\ 4\ 2\ 2\ 1 \\ 4\ 1\ 1\ 1 \\ 2\ 1 \\ 2\ 1 \\ 1 \end{array}$$

5 4 2 2 1 is a self-conjugate partition of the number 14,
2 1 „ „ „ „ „ 3,

the corresponding symmetrical two-dimensional graphs being

$$\begin{array}{lll} \bullet\ \bullet\ \bullet\ \bullet\ \bullet & & \bullet\ \bullet \\ \bullet\ \bullet\ \bullet\ \bullet & & \bullet \\ \bullet\ \bullet & & \\ \bullet\ \bullet & & \\ \bullet & & \end{array}$$

Hence this self-conjugate property appertains also to the lateral appendage, the lines of numbers being taken parallel to the axis of $y$, *not* parallel to the axis of $x$. The reverse would naturally be the case if we were considering the subjacent appendage. This property imposes a limitation upon the possible forms of lateral appendage.

Let $w_1$ be the content of the base, $i_1$ its order i.e. the number of nodes along an axis; also let $w_2$ and $i_2$ refer to the lateral.

Then for the complete symmetrical graph we have content $w_1 + 3w_2$ and order $i_1 + i_2$, or say, $w$, $i$ referring to the complete graph,

$$w = w_1 + 3w_2, \quad i = i_1 + i_2.$$

For the base 1

$w_1 = 1$, $i_1 = 1$ the lateral must have the form

$$1\ 1\ 1\ 1 \ldots,$$

and the generating function for such laterals whose order does not exceed $i_2$ is

$$\frac{1 - x^{i_2+1}}{1 - x};$$

therefore if $F(x)$ denote the generating function of the associated symmetrical graphs

$$^{*}\mathrm{Co}\, x^{w} F(x) = \mathrm{Co}\, x^{1+2w_2} F(x) = \mathrm{Co}\, x^{w_2} \frac{1-x^{i_2+1}}{1-x},$$

$$\therefore\ F(x) = x\,\frac{1-x^{2i_2+2}}{1-x^2},$$

the generating function of symmetrical graphs on the base 1.

Since $i_2 = i-1$, we may write this

$$F(x) = x\,\frac{1-x^{2i}}{1-x^2}.$$

For the base $\begin{matrix}2&2\\2&1\end{matrix}$ the lateral may involve $\begin{matrix}2\\1\end{matrix}$ and 1 but not $\begin{matrix}1\\1\end{matrix}$; hence its form must be

$$\begin{matrix}2\ 2\ 2 \ldots 1\ 1\ 1 \ldots\\ 1\ 1\ 1\end{matrix}$$

If $i_2$ be unrestricted the lateral generating function is

$$\frac{1}{1-x\,.\,1-x^2};$$

otherwise we have to seek the coefficient of $m^{i_2}x^{w_2}$ in

$$\frac{1}{1-m\,.\,1-mx\,.\,1-mx^2};$$

and since $\qquad w_1 = 7,\ i_1 = 2,\ i_2 = i-2,$

we obtain the generating function of symmetrical graphs

$$\frac{x^7}{1-m\,.\,1-mx^2\,.\,1-mx^6},$$

in which we seek the coefficient of $m^{i-2}x^{w}$.

Similarly for the base $\begin{matrix}2&2\\2&2\end{matrix}$ since the lateral must be of the form $\begin{matrix}2\ 2 \ldots 2\ 2 \ldots 1\ 1 \ldots\\ 2\ 2 \ldots 1\ 1 \ldots\end{matrix}$ we are led to the generating function

$$\frac{x^8}{1-m\,.\,1-mx^3\,.\,1-mx^6\,.\,1-mx^{12}},$$

in which we seek the coefficient of $m^{i-2}x^{w}$.

If the base is at most of order 2 we may say that the enumeration of symmetrical graphs of content $w$ and having at most $i$ nodes along an axis is given by the coefficient of $m^0x^w$ in

$$1 + \frac{m^{1-i}x}{1-m\,.\,1-mx^2} + \frac{m^{2-i}x^7}{1-m\,.\,1-mx^3\,.\,1-mx^6} + \frac{m^{2-i}x^8}{1-m\,.\,1-mx^3\,.\,1-mx^6\,.\,1-mx^{12}}.$$

* $\mathrm{Co}\, x^{w} F(x)$ denotes the coefficient of $x^{w}$ in the expansion of $F(x)$.

If $i_2$ and therefore $i$ be unrestricted this expression naturally becomes

$$1+\frac{x}{1-x^3}+\frac{x^7}{1-x^3.1-x^9}+\frac{x^8}{1-x^3.1-x^9.1-x^{12}}.$$

The question now arises as to the direct formation of the fractions appertaining to the bases of order $i_1$.

The form of the lateral appendage depends, as we have seen, upon the self-conjugate unipartite partition represented by the right-hand column or boundary of the base. We will call this partition the base-lateral. So far of the first four orders we have met with certain base-laterals, viz.:—

| Order $i_1=$ | Base-lateral | | | |
|---|---|---|---|---|
| 1 | 1 | | | |
| 2 | 2 | 2 | | |
| | 1 | 2 | | |
| 3 | 3 | 3 | 3 | |
| | 1 | 3 | 3 | |
| | 1 | 2 | 3 | |
| 4 | 4 | 4 | 4 | 4 |
| | 1 | 4 | 4 | 4 |
| | 1 | 2 | 4 | 4 |
| | 1 | 2 | 3 | 4 |

Of order $i_1$ there are $i_1$ different base-laterals; for consider the formation of the base of order $n$ from those of inferior orders. Combination of the half-hollow-cube form of order $i_1$ with the bases of orders less than $i_1-1$ can only result in base-laterals identical with that of the half-hollow-cube base; and assuming that base-laterals of order $i_1-1$ are $i_1-1$ in number, it is plain that the combination referred to can only produce $i_1-1$ additional base-laterals. Hence, on the assumption made, the whole number of base-laterals of order $i_1$ is $1+i_1-1=i_1$. By induction the theorem is established.

The $i_1$ base-laterals of order $i_1$ are (writing them for convenience horizontally instead of vertically)

$$i_1 1^{i_1-1},\quad i_1{}^2 2^{i_1-2},\quad i_1{}^3 3^{i_1-3},\ \ldots\ldots\ i_1{}^{i_1}.$$

We must discover the generating function of bases having a given base-lateral $i_1{}^s s^{i_1-s}$.

The base-lateral in question is associated with $2^{i_1-s-1}$ different bases if $s<n$, while $i_1{}^{i_1}$ is associated with but a single base. Taking $s=1$, the simplest base, having $i_1 1^{i_1-1}$ for base-lateral, is the half-hollow-graph of content $i_1{}^2-(i_1-1)^2$. The remaining bases with this base-lateral are obtained by combination with the bases of the first $i_1-2$ orders.

Denote by $u_s - 1$ the generating function of the bases of the first $s$ orders; then

$$u_{s+1} = \{1 + x^{(s+1)^2 - s^2}\} u_s,$$

or

$$u_{i_1-2} = \{1 + x^{(i_1-2)^2-(i_1-3)^2}\} u_{i_1-3},$$

whence

$$u_{i_1-2} = (1+x)(1+x^{3^2-1^2})(1+x^{5^2-2^2})\dots\{1+x^{(i_1-2)^2-(i_1-3)^2}\}.$$

Therefore the generating function of bases, having the base-lateral $i_1 1^{i_1-1}$, is

$$x^{i_1^2-(i_1-1)^2}[(1+x)(1+x^{3^2-1^2})(1+x^{5^2-2^2})\dots\{1+x^{(i_1-2)^2-(i_1-3)^2}\}].$$

Next consider the bases having the base-lateral

$$i_1^2 2^{i_1-2}.$$

The simplest base of this nature is derived by combining the half-hollow-cube form of content $i_1^3-(i_1-1)^3$ with the similar form of content $(i_1-1)^3-(i_1-2)^3$ and thus it has the content $i_1^3-(i_1-2)^3$. With this we can again combine every base of the first $i_1-3$ orders without altering the base-lateral, which remains $i_1^2 2^{i_1-2}$. Hence the bases are enumerated by the generating function

$$x^{i_1^2-(i_1-2)^2}[(1+x)(1+x^{3^2-1^2})(1+x^{5^2-2^2})\dots\{1+x^{(i_1-3)^2-(i_1-4)^2}\}].$$

In general the simplest base with base-lateral $i_1^s s^{i_1-s}$ is obtained by combining the half-hollow-cube forms of orders $i_1$, $i_1-1$, $i_1-2, \dots i_1-s+1$, and thus has the content

$$i_1^3-(i_1-s)^3.$$

By reasoning before employed we arrive at the fact that the bases with base-lateral $i_1^s s^{i_1-s}$ are enumerated by

$$x^{i_1^2-(i_1-s)^2}[(1+x)(1+x^{3^2-1^2})(1+x^{5^2-2^2})\dots\{1+x^{(i_1-s-1)^2-(i_1-s-2)^2}\}].$$

In this expression, between the brackets [ ] there are $i_1-s-1$ factors; if $s=i_1-1$ or $i_1$ we take merely $x^{i_1^2-1^2}$ and $x^{i_1^2}$ respectively.

The next question is the ascertainment of the generating function which enumerates the lateral appendages that can be associated with the base-lateral $i_1^s s^{i_1-s}$. When $s=1$ this is easy because the lateral must be composed of columns

$$\begin{array}{cccc} 1 & 2 & 3 \dots & i_1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \\ & & & \vdots \end{array}$$

not more than $i_2$ being taken.

The generating function is

$$\frac{1}{1-m\,.\,1-mx\,.\,1-mx^3\,.\,1-mx^5\,.\dots.\,1-mx^{2i_1-1}},$$

where we seek the coefficient of $m^{i_2}x^{w_2}$ or of $m^{i-i_1}x^{w_2}$ since $i=i_1+i_2$.

Finally for the symmetrical graphs constructed on bases whose base-laterals are $i_1 1^{i_1-1}$ we have the generating function

$$\frac{x^{i_1^2-(i_1-1)^2}\left[(1+x)(1+x^{2^2-1^2})(1+x^{3^2-2^2})\dots\{1+x^{(i_1-2)^2-(i_1-3)^2}\}\right]}{(1-m)(1-mx^3)(1-mx^9)\dots(1-mx^{3i_1-3})},$$

in the expansion of which we must seek the coefficient of $m^{i-i_1}x^{w_0}$.

When the base-lateral is $i_1^s s^{i_1-s}$ the matter is by no means so simple. The lateral appendage is composed of columns each of which is a self-conjugate partition of a number, and the possible forms of the columns are further limited by the form of the base-lateral. To explain take $i_1 = 5, s = 3$ so that the base-lateral (written horizontally) is

5 5 5 3 3.

This has a graph

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \end{array}$$

formed of three plane angles. Any column of the lateral must have a graph which can be superposed; the condition for this is obviously that it must be composed of not more than three plane angles, the largest angle containing not more than 9 nodes. So with base-lateral $i_1^s s^{i_1-s}$ a lateral column must have a graph composed of not more than $s$ plane angles, the largest angle containing not more than $2i_1 - 1$ nodes.

The complete lateral appendage constitutes a multipartite partition whose graph is symmetrical in two dimensions. We have therefore to enumerate the graphs of this nature, each layer being composed of at most $s$ plane angles and no angle containing more than $2i_1 - 1$ nodes and the number of layers not exceeding $i - i_1$ or $i_2$.

The crude form of this generating function was found earlier in the paper to be

$$\Omega\frac{1}{(1-m)(1-mx)(1-max^3)(1-mabx^5)\dots(1-mabc\dots x^{2i_1-1})},$$

$$\left(1-\frac{x}{a}\right)\left(1-\frac{a'}{ab}x^3\right)\left(1-\frac{a'b'}{abc}x^5\right)\dots \text{ ad inf.},$$

$$\left(1-\frac{x}{a'}\right)\left(1-\frac{a''}{a'b'}x^3\right)\left(1-\frac{a''b''}{a'b'c'}x^5\right)\dots \text{ ad inf.},$$

.....

$s$ rows,

in which we take the coefficient of $m^{i-i_1}x^{w_2}$.

We can now assert that the symmetrical graphs which appertain to the base-lateral $i_1^s s^{i_1-s}$, and have at most $i$ nodes along an axis, are enumerated by the coefficient of $m^{i-i_1}x^w$ in

$$\Omega \frac{x^{i_1^2-(i_1-s)^2}\left[(1+x)(1+x^{2^2-1^2})(1+x^{3^2-2^2})\ldots\{1+x^{(i_1-s-1)^2-(i_1-s-2)^2}\}\right]}{(1-m)(1-mx^3)(1-max^9)(1-mabx^{15})\ldots(1-mabc\ldots x^{6i_1-3})},$$

$$\left(1-\frac{x^3}{a}\right)\left(1-\frac{a'}{ab}x^9\right)\left(1-\frac{a'b'}{abc}x^{15}\right)\ldots \text{ad inf.},$$

$$\left(1-\frac{x^3}{a'}\right)\left(1-\frac{a''}{a'b'}x^9\right)\left(1-\frac{a''b''}{a'b'c'}x^{15}\right)\ldots \text{ad inf.},$$

.....

$s$ rows.

Denoting this expression by $S(m, x)$, we see that

$$\sum_{i_1=1}^{i_1=i}\sum_{s=1}^{s=i_1} S(m, x) \,.\, m^{i_1-i}$$

enumerates, by the coefficient of $m^0x^w$, the whole of the symmetrical graphs subject to the single restriction that more than $i$ nodes are not to occur along an axis.

# TRANSACTIONS

OF THE

# CAMBRIDGE

# PHILOSOPHICAL SOCIETY.

---

VOLUME XVII. PART III.

CAMBRIDGE:
AT THE UNIVERSITY PRESS

M.DCCC.XCIX.

## ADVERTISEMENT.

---

*The Society as a body is not to be considered responsible for any facts and opinions advanced in the several Papers, which must rest entirely on the credit of their respective Authors.*

---

The Society takes this opportunity of expressing its grateful acknowledgments to the Syndics of the University Press for their liberality in taking upon themselves the expense of printing this Part of the Transactions.

# CONTENTS.

PAGE

VI. *On Divergent (or Semiconvergent) Hypergeometric Series.* By Professor W. McF. Orr, M.A., Royal College of Science, Dublin ..... 171

VII. *A semi-inverse method of solution of the equations of elasticity, and its application to certain cases of aeolotropic ellipsoids and cylinders.* By C. Chree, Sc.D., F.R.S. ... 201

VIII. *On the Change of a System of Independent Variables.* By E. G. Gallop, M.A., Fellow of Gonville and Caius College..... 231

IX. *On Divergent Hypergeometric Series.* By Professor W. McF. Orr, M.A. ..... ..... 283

# VI. *On Divergent (or Semiconvergent) Hypergeometric Series.*

By Prof. W. McF. Orr, M.A., Royal College of Science, Dublin.

[*Received* and *read* May 16, *Revised* September 1898.]

1. The series

$$1+\frac{\alpha_1\alpha_2\ldots\alpha_m}{1.\rho_1\rho_2\ldots\rho_n}x+\frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)\ldots\alpha_m(\alpha_m+1)}{1.2.\rho_1(\rho_1+1)\rho_2(\rho_2+1)\ldots\rho_n(\rho_n+1)}x^2+\ldots\ldots\ldots\ldots(1)$$

is convergent for all values of $x$ if $m \not> n$ and convergent for values of $x$ whose modulus is less than unity if $m=n+1$; in such cases if its sum be denoted by

$$F(\alpha_1,\ \alpha_2,\ldots\alpha_m;\ \rho_1,\ \rho_2,\ldots\rho_n;\ x)$$

the successive differential coefficients of this expression are represented numerically by the convergent series obtained by taking the corresponding differential coefficients of the terms of (1). The relation connecting the coefficients of two consecutive terms $a_rx^r$, $a_{r+1}x^{r+1}$, viz. :—

$$(1+r)(\rho_1+r)\ldots(\rho_n+r)a_{r+1}=(\alpha_1+r)(\alpha_2+r)\ldots(\alpha_m+r)a_r\ldots\ldots\ldots\ldots\ldots(2),$$

is equivalent to the differential equation

$$\left\{(\theta+\alpha_1)(\theta+\alpha_2)\ldots(\theta+\alpha_m)-\frac{1}{x}\theta(\theta+\rho_1-1)(\theta+\rho_2-1)\ldots(\theta+\rho_n-1)\right\}y=0\ \ldots(3),$$

in which $\theta$ stands for the operator $xd/dx$. The series (1) is therefore, when convergent, a solution of this equation. Relation (2) is satisfied by $n$ other series, convergent if (1) be convergent, one of which is

$$x^{1-\rho_1}F(\alpha_1-\rho_1+1,\ \alpha_2-\rho_1+1,\ldots\alpha_m-\rho_1+1;\ 2-\rho_1,\ \rho_2-\rho_1+1,\ldots\rho_n-\rho_1+1;\ x)\ldots(4),$$

the others being analogous. Each of these $n$ series when convergent is therefore a solution of equation (3), and the $n+1$ series thus furnish the complete solution of this equation for all values of $x$ if $m \not> n$, and for values of $x$ whose modulus is less than unity if $m=n+1$. It is supposed that no two of the quantities $\alpha_1,\ldots\alpha_m$, $\rho_1,\ldots\rho_n$, 1 are equal, or differ by an integer.

Relation (2) is also satisfied by $m$ series proceeding in descending powers of $x$, one of which is

$$x^{-\alpha_1}\left\{1+\frac{\alpha_1(\alpha_1-\rho_1+1)\ldots(\alpha_1-\rho_n+1)}{1.(\alpha_1-\alpha_2+1)\ldots(\alpha_1-\alpha_m+1)}(-)^{n-m+1}\frac{1}{x}\right.$$
$$\left.+\frac{\alpha_1(\alpha_1+1)(\alpha_1-\rho_1+1)(\alpha_1-\rho_1+2)\ldots(\alpha_1-\rho_n+1)(\alpha_1-\rho_n+2)}{1.2.(\alpha_1-\alpha_2+1)(\alpha_1-\alpha_2+2)\ldots(\alpha_1-\alpha_m+1)(\alpha_1-\alpha_m+2)}\frac{1}{x^2}+\ldots\right\}\ldots(5),$$

the others being analogous. These $m$ series are all divergent when the former $n+1$ are convergent. The object of the present paper is to show that in such a case if the real part of $(-1)^{n-m+1}/x$ is negative, then provided $s$ exceeds a certain number independent of $x$, the sum of the first $s$ terms of such a series as (6) differs by some quantity whose modulus is less than that of the next term from a certain linear function of the $n+1$ convergent series (1), (5), etc., and that whether the real part of $(-1)^{n-m+1}x$ is positive or negative, for any specified value of $s$, $x$ can be taken so great that the sum of the first $s$ terms differs from the same linear function of the convergent series by a quantity whose modulus is less than that of the next term multiplied by $1+\epsilon$, where $\epsilon$ is any assigned small positive quantity.

It may be remarked that the theorem stated for the case in which $(-)^{n-m+1}x$ has its real part negative *cannot* be true without *some* restriction on the argument or the modulus of $(-)^{n-m+1}x$. It has been pointed out by Hankel (*Math. Annal.* Vol. I.) that such a theorem cannot hold for a series proceeding in powers of $u$ whose terms, after a certain one, are real positive and increasing as the three results to which it would lead on terminating the series successively before each of three consecutive positive terms, of which the first is less than the second and the second less than the third, involve an inconsistency. Hankel however appears to consider that it may hold for all other values of $u$; this is a mistake unless the function of $u$ to which the semiconvergent series is "equal" (in the sense above) is discontinuous on crossing some curve other than the positive part of the axis of real quantities, as it is evident that, with this exception, if the theorem be true for all save real and positive values of $u$ it must be true even for these.

The well-known semiconvergent expansions of $J_n(x)$, for example, *cannot* therefore hold for complex values in the sense that for each of the two divergent series occurring therein the error committed in stopping after the $s$th term, provided $s-n+\frac{1}{2}$ is positive, has a modulus less than that of the next term, without some restriction on the value of $x$; and in fact the demonstrations given by Lipschitz (*Crelle*, LVI.), Hankel (*loc. cit.*), and Gray and Mathews (*Treatise on Bessel Functions*) are invalid unless $x$ be wholly real. Lipschitz, who discusses only the case in which $n$ is zero, appears in fact to consider only real values of $x$. The fallacies involved in the proofs by Hankel and by Gray and Mathews will be noted presently. The magnitude of the error in case $x$ is complex has been discussed by H. Weber (*Math. Annal.* Vol. XXXVII.), who has not however explicitly referred to the fallacies in question.

2. As a lemma to be used in establishing the theorem of the present paper I proceed to prove that if $(1-t)^{-\alpha}$ be expanded in ascending powers of $t$, where the modulus of $t$ may be greater than unity, then provided $\alpha+s$ be positive the modulus of the error outstanding after $s$ terms is, if the real part of $t$ be negative, less than that of the next term, and if the real part of $t$ be positive, and $t$ not wholly real, less than that of a certain multiple (involving the argument of $t$) of the next term.

We have

$$\frac{1}{z-t}=\frac{1}{z}\left\{1+\frac{t}{z}+\frac{t^2}{z^2}+\dots\frac{t^{s-1}}{z^{s-1}}\right\}+\frac{t^s}{z^s(z-t)}.$$

Multiplying by $\frac{(1-z)^{-\alpha}dz}{2\pi i}$, where $(1-z)^{-\alpha}$ is equal to unity at the origin, and integrating along the path *ABCDEFA* in Fig. 1, *CDEFA* being supposed a curve, every point of which is at an infinite distance from the origin, we obtain

$$(1-t)^{-\alpha} = 1 + \alpha t + \ldots \text{(to } s \text{ terms)} + \frac{t^s}{2\pi i}\int \frac{(1-z)^{-\alpha}}{z^s(z-t)}dz,$$

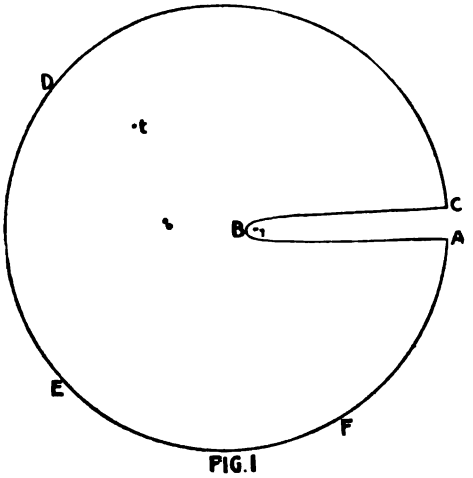


FIG. I

wherein the path of integration is the same and $(1-t)^{-\alpha}$ reduces to unity at the origin. If now $\alpha + s$ is positive the path *CDEFA* contributes nothing to the integral on the right and the remainder after $s$ terms, which we will denote by $R(\alpha, s)$, is equal to the integral taken along the path *ABC* only. This may be written, following Pochhammer (*Math. Annal.* XXXV.), in the form

$$\frac{-t^s}{2\pi i}\int_{\infty}^{(1)} \frac{(1-z)^{-\alpha}}{z^s(z-t)}dz.$$

Suppose first that $1-\alpha$ is positive. We then have

$$R(\alpha, s) = \frac{t^s}{2\pi i}(e^{\alpha\pi i} - e^{-\alpha\pi i})\int_1^\infty \frac{(z-1)^{-\alpha}}{z^s(z-t)}dz$$

$$= \frac{t^s}{\pi}\sin\alpha\pi\int_1^\infty \frac{(z-1)^{-\alpha}}{z^s(z-t)}dz.$$

If the real part of $t$ be negative

$$\text{mod.}\int_1^\infty \frac{(z-1)^{-\alpha}}{z^s(z-t)}dz < \int_1^\infty \frac{(z-1)^{-\alpha}}{z^{s+1}}dz < \frac{\Pi(-\alpha)\,\Pi(s+\alpha-1)}{\Pi(s)}.$$

Therefore since $\Pi(\alpha-1)\,\Pi(-\alpha) = \pi \operatorname{cosec}\alpha\pi$ we have

$$\text{mod. } R(\alpha, s) < \text{mod.}\frac{t^s\Pi(s+\alpha-1)}{\Pi(s)\Pi(\alpha-1)},$$

i.e. < modulus of first term omitted as stated.

If the real part of $t$ be positive and $t = r(\cos\phi + i\sin\phi)$,

$$\text{mod.} \int_1^\infty \frac{(z-1)^{-\alpha}}{z^s(z-t)} dz < \frac{1}{\sin\phi} \int_1^\infty \frac{(z-1)^{-\alpha}}{z^{s+1}} dz,$$

and therefore

$$\text{mod. } R(\alpha, s) < t^s \frac{\Pi(s+\alpha-1)}{\Pi(s)\,\Pi(\alpha-1)} \frac{1}{\sin\phi}.$$

Next suppose that $1-\alpha$ is negative but that $n+1-\alpha$ is positive, $n$ being a positive integer. By partial integration performed $n$ times in succession we obtain the result

$$R(\alpha, s) = \frac{(-)^{n+1} t^s}{2\pi i.(1-\alpha)(2-\alpha)\dots(n-\alpha)} \int_\infty^{(1)} dz.(1-z)^{n-\alpha} \frac{d^n}{dz^n}\left\{\frac{1}{z^s(z-t)}\right\}$$

$$= \frac{-\Pi(n)\,t^s}{2\pi i(1-\alpha)(2-\alpha)\dots(n-\alpha)} \int_\infty^{(1)} (1-z)^{n-\alpha} \left\{\frac{1}{z^s(z-t)^{n+1}} + \frac{s}{1.z^{s+1}(z-t)^n} + \frac{s.(s+1)}{1.2.z^{s+2}(z-t)^{n-1}}\right.$$

$$\left. + \dots + \frac{s(s+1)\dots(s+n-1)}{\Pi(n).z^{s+n}(z-t)}\right\} dz$$

$$= \frac{\Pi(n)\,t^s \sin(\alpha-n)\pi}{\pi.(1-\alpha)(2-\alpha)\dots(n-\alpha)} \int_1^\infty (z-1)^{n-\alpha} \left\{\frac{1}{z^s(z-t)^{n+1}} + \frac{s}{1.z^{s+1}(z-t)^n} + \dots\right.$$

$$\left. + \frac{s(s+1)\dots(s+n-1)}{\Pi(n)\,z^{s+n}(z-t)}\right\} dz \quad \dots\dots(6).$$

If now the real part of $t$ be negative, this integral would be increased in absolute value by replacing every negative power of $z-t$ by the same power of $z$; if this were done the coefficient of $(z-1)^{n-\alpha} z^{-s-n-1}$ under the sign of integration would be

$$1 + \frac{s}{1} + \frac{s(s+1)}{1.2} + \dots \frac{s(s+1)\dots(s+n-1)}{1.2\dots n},$$

which is equal to

$$\frac{(s+1)(s+2)\dots(s+n)}{1.2\dots n}.$$

Accordingly, since

$$\int_1^\infty (z-1)^{n-\alpha} z^{-s-n-1} dz = \frac{\Pi(n-\alpha)\,\Pi(s+\alpha-1)}{\Pi(s+n)},$$

we have

$$\text{mod. } R(\alpha, s) < \text{mod. } \frac{t^s \sin(\alpha-n)\pi.\Pi(s+\alpha-1)\,\Pi(-\alpha)}{\pi.\Pi(s)}$$

$$< \text{mod. } \frac{t^s \Pi(s+\alpha-1)}{\Pi(s)\,\Pi(\alpha-1)},$$

i.e. < modulus of next term as stated *.

If the real part of $t$ be positive and $t = r(\cos\phi + i\sin\phi)$ the integral in (6) would be increased in absolute value by replacing every negative power of $z-t$ by the same power of $z\sin\phi$, and we evidently can deduce that

$$\text{mod. } R(\alpha, s) < \text{modulus of the next term multiplied by } \operatorname{cosec}^{n+1}\phi.$$

* This result follows readily also from Lagrange's form of the remainder in Taylor's series, extended (as regards the modulus) to a function of a complex variable. See Darboux, *Liouville's Journal*, 1876.

3. In discussing the functions of Art. 1 it will be sufficient to consider the case in which $m=n$. The case in which $m=n+1$ can be derived from this by making $x$ and one of the $\rho$'s infinite together, and any case in which $m<n$ by making one or more of the $\alpha$'s infinite and $x$ infinitely small and of the proper order.

Omitting the case of $n=0$, the simplest of such cases is that in which $n=1$; this case will be considered somewhat fully. Equation (3) is now reduced to

$$\{xD^2-(x-\rho)D-\alpha\}y=0 \quad\text{.....}\quad (7),$$

$D$ denoting differentiation with respect to $x$. Its solutions in converging series have been fully discussed by Pochhammer (*Math. Annalen*, Vol. XXXVI.).

If the series $F(\alpha;\ \rho;\ x)$ be considered merely as the limiting form of that for

$$F(\alpha,\ \beta;\ \rho;\ u)$$

wherein $x/\beta$ is written for $u$, and $\beta$ is then made infinite, the limiting form of the equation

$$F(\alpha,\ \beta;\ \rho;\ u)=(1-u)^{\rho-\alpha-\beta}F(\rho-\alpha,\ \rho-\beta;\ \rho;\ u)$$

shows that

$$F(\alpha;\ \rho;\ x)=e^x F(\rho-\alpha;\ \rho;\ -x) \quad\text{.....}\quad (8),$$

and that

$$x^{1-\rho}F(\alpha-\rho+1;\ 2-\rho;\ x)=x^{1-\rho}e^x F(1-\alpha;\ 2-\rho;\ -x) \quad\text{.....}\quad (9),$$

while another particular integral suggests another divergent series, viz.

$$e^x x^{\alpha-\rho}\left\{1+\frac{(1-\alpha)(\rho-\alpha)}{1}\frac{1}{x}+\frac{(1-\alpha)(2-\alpha)(\rho-\alpha)(\rho-\alpha+1)}{1.2}\frac{1}{x^2}+\dots\right\} \quad\text{.....}\quad (10),$$

besides that of the type (5) which now is

$$x^{-\alpha}\left\{1-\frac{\alpha(\alpha-\rho+1)}{1}\frac{1}{x}+\frac{\alpha(\alpha+1)(\alpha-\rho+1)(\alpha-\rho+2)}{1.2}\frac{1}{x^2}+\dots\right\} \quad\text{.....}\quad (11).$$

A particular case of relation (8), in which $\rho$ is written $=2\alpha=n$ and $x$ is replaced by $2x$, is given by Glaisher (*Trans. Roy. Soc.*, 1881, Part 3, page 774).

Pochhammer shows that equation (7) is satisfied by the integral

$$\int e^u(u-x)^{-\alpha}u^{\alpha-\rho}du$$

taken along a path which starts from any part and returns to the same point, provided the path is such that the initial and final values of $e^u(u-x)^{-\alpha}u^{\alpha-\rho}$ differ by zero, and has considered the two solutions

$$\int_c^{\overline{(x,0,x-,0-)}} e^u(u-x)^{-\alpha}u^{\alpha-\rho}du \quad\text{.....}\quad (12),$$

$$\int_{-\infty}^{\overline{(x,0)}} e^u(u-x)^{-\alpha}u^{\alpha-\rho}du \quad\text{.....}\quad (13).$$

In the former the path starts from any arbitrary point $c$ (which may be taken in the finite line joining the points 0, $x$), and returns to the same point after having made a circuit round the points $x$, 0, in the positive direction, and then a circuit round the points $x$, 0 in

the negative direction; this integral is a multiple of $x^{1-\rho}F(\alpha-\rho+1;\ 2-\rho;\ x)$. In the latter the path starts from the point $-\infty$ and returns to the same point after making a circuit round the points $x$, 0, in the positive direction; this integral is a multiple of $F(\alpha;\ \rho;\ x)$.

The identities (8), (9) might easily be established from these integrals by making the substitution $u = x - v$.

In addition to (12), (13) the following integrals, not discussed by Pochhammer, also satisfy the necessary conditions and are therefore solutions of equation (7), viz.

$$\int_{-\infty}^{(0)} e^u (u-x)^{-\alpha} u^{\alpha-\rho} du \quad \text{.....(14)},$$

$$\int_{-\infty}^{(x)} e^u (u-x)^{-\alpha} u^{\alpha-\rho} du \quad \text{.....(15)}.$$

We take the path of $u$ in (15), both in going and returning, to pass above or below the origin according as $x$ is above or below the axis of real quantities.

Suppose in the first instance that the real part of $x$ is positive.

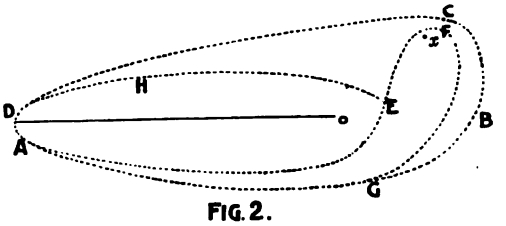



FIG. 2.

In Fig. 2 let $A$ be a point near the negative part of the axis of real quantities and at a great distance from the origin. The path of $u$ in (12) is equivalent to the paths $AGBCDA$, $AEFGA$, $ADHEA$, the value of the function $e^u(u-x)^{-\alpha}u^{\alpha-\rho}$ at the end of one part being taken as the value at the beginning of the next part. But for the path $AEFGA$ we may substitute $AEHDA$, $ADCBGA$. Thus for the whole path in (12) we may take $AGBCDA$, $AEHDA$, $ADCBGA$, $ADHEA$. And as the distance of $A$ from the origin increases indefinitely, the first of these becomes the path in (13), the third this reversed, the second the path in (14), and the last that path reversed *.

In the integral (12) writing $(u-x)^{-\alpha}u^{\alpha-\rho} = x^{-\rho}\left(\frac{u}{x}-1\right)^{-\alpha}\left(\frac{u}{x}\right)^{\alpha-\rho}$, we will choose as Pochhammer virtually does those values of $\left(\frac{u}{x}-1\right)^{-\alpha}$ and of $\left(\frac{u}{x}\right)^{\alpha-\rho}$ which, when the path of $u$ crosses the production of the line joining the points 0, $x$, for the first time at $C$, are wholly real and positive, and that value of $x^{-\rho}$ whose argument lies between $-\rho\pi/2$, and $+\rho\pi/2$. Pochhammer, *loc. cit.*, shows that this integral is

$$e^{\pi i(\alpha-\rho)}\,\mathfrak{E}(\alpha-\rho+1,\ 1-\alpha)\,x^{1-\rho}F(\alpha-\rho+1;\ 2-\rho;\ x) \quad \text{.....(16)},$$

where

$$\mathfrak{E}(a,\ b) = e^{-\pi i(a+b)}\int_{c}^{(1,\,0,\,1-,\,0-)} v^{a-1}(1-v)^{b-1}\,dv \quad \text{.....(17)},$$

* On some grounds it would be more convenient to choose as the initial and final point one near the *positive* part of the axis of real quantities and at a great distance from the origin. I thought it advisable in the present case to follow Pochhammer.

$c$ being an arbitrary point on the finite line joining the points 0, 1 and $v^{a-1}$, $(1-v)^{b-1}$ having there initially values which are real and positive. (See Pochhammer, "Zur Theorie der Euler'schen Integrale," *Math. Annal.* Vol. XXXV.)

But $$\mathfrak{E}(a, b) = -4 \sin a\pi \sin b\pi \frac{\Pi(a-1)\Pi(b-1)}{\Pi(a+b-1)},$$

the $\Pi$ functions having their extended meaning. The integral (12) is thus equal to

$$-4e^{\pi i(\alpha-\rho)} \sin(\alpha-\rho)\pi \sin(-\alpha\pi) \frac{\Pi(\alpha-\rho)\Pi(-\alpha)}{\Pi(1-\rho)} x^{1-\rho} F(\alpha-\rho+1;\ 2-\rho;\ x) \ldots(18).$$

Also of the four portions by which we have shown the path in (12) can be replaced, the first $AGBCDA$ contributes to the integral as shown by Pochhammer,

$$\bar{\Gamma}(1-\rho) F(\alpha;\ \rho;\ x) \text{ or } \frac{2\pi i}{\Pi(\rho-1)} F(\alpha;\ \rho;\ x) \ldots\ldots\ldots\ldots\ldots\ldots(19),$$

and the third $ADCBGA$ contributes this same multiplied by $-e^{2\pi i(\alpha-\rho)}$. These two together then contribute

$$\frac{4\pi e^{\pi i(\alpha-\rho)} \sin(\alpha-\rho)\pi}{\Pi(\rho-1)} F(\alpha;\ \rho;\ x) \ldots\ldots\ldots\ldots\ldots\ldots(20).$$

Again for the integral along the second portion $AEHDA$, bearing in mind the alteration of the argument, we have

$$e^{\pi i(\alpha-2\rho)} x^{-\alpha} \int_{-\infty}^{\bar{(0)}} e^u (1-u/x)^{-\alpha} u^{\alpha-\rho} du \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

wherein at the point where the path of $u$ crosses the line $0, x$, the values of $(1-u/x)^{-\alpha}$ and of $(u/x)^{\alpha-\rho}$ are real and positive and the argument of $x^{-\alpha}$ lies between $-\alpha\pi/2$ and $+\alpha\pi/2$.

Writing $$(1-u/x)^{-\alpha} = 1 + \frac{\alpha}{1}\frac{u}{x} + \frac{\alpha(\alpha+1)}{1.2}\frac{u^2}{x^2} + \ldots \text{ (to } s \text{ terms)} + R_s,$$

and evaluating the several terms of the integral, (21) becomes

$$e^{\pi i(\alpha-2\rho)} \{e^{\pi i(\rho-\alpha)} - e^{\pi i(\alpha-\rho)}\} \Pi(\alpha-\rho) x^{-\alpha}$$
$$\left\{1 - \frac{\alpha(\alpha-\rho+1)}{1}\frac{1}{x} + \frac{\alpha(\alpha+1)(\alpha-\rho+1)(\alpha-\rho+2)}{1.2}\frac{1}{x^2} + \ldots \text{ to } s \text{ terms} + R_s'\right\} \ldots(22),$$

where $$R_s' = \int_{-\infty}^{\bar{(0)}} e^u u^{\alpha-\rho} R_s du \div \int_{-\infty}^{\bar{(0)}} e^u u^{\alpha-\rho} du \ldots\ldots\ldots\ldots\ldots\ldots(23).$$

Now the origin of $u$ is a multiple point of order $s$ on $R_s$, or $R_s$ in the neighbourhood of the origin of $u$ is of order $u^s$, as may be seen by finding the limiting value of $R_s/u^s$, and therefore provided $\alpha-\rho+s+1$ is positive,

$$\int_{-\infty}^{\bar{(0)}} e^u u^{\alpha-\rho} R_s du = -\left(e^{\pi i(\alpha-\rho+s)} - e^{-\pi i(\alpha-\rho+s)}\right) \int_{-\infty}^{0} e^u u^{\alpha-\rho} R_s du \ldots\ldots\ldots\ldots(24),$$

(the right-hand member involving a line integral and the argument of $e^u u^{\alpha-\rho} R_s$ being the same as that of $R_s/u^s$). And if $\alpha+s$ is positive the modulus of $R_s$ is less than that of the $(s+1)$th term in the expansion of $(1-u/x)^{-\alpha}$, accordingly $R_s'$ is less in absolute value than the $(s+1)$th term of the series in (22).

In this sense then provided $\alpha+s$, $\alpha-\rho+s+1$ and the real part of $x$ are positive, $s$ terms of the series represent the whole of the expression, including $R_s'$, in brackets in (22), an expression which we will denote by $\phi(\alpha, \alpha-\rho+1; -1/x)$.

The fourth and last portion of the path in (12) contributes to (12) an expression equal to (22) multiplied by $-e^{2\pi i\rho}$.

Thus the second and last portions together contribute

$$4e^{\pi i(\alpha-\rho)} \sin\rho\pi \sin(\rho-\alpha)\pi \,.\, \Pi(\alpha-\rho)\, x^{-\alpha}\phi(\alpha, \alpha-\rho+1; -1/x) \quad\text{.....(25).}$$

Equating (18) to the sum of (20) and (25) an equation is obtained which may be written in the form

$$\Pi(\alpha-1)\,\Pi(-\rho)\,F(\alpha;\rho;x) + \Pi(\alpha-\rho)\,\Pi(\rho-2)\,x^{1-\rho}F(\alpha-\rho+1; 2-\rho; x)$$
$$= \Pi(\alpha-1)\,\Pi(\alpha-\rho)\,x^{-\alpha}\phi(\alpha, \alpha-\rho+1; -1/x) \quad\text{.....(26),}$$

the argument of $x$ lying between $-\pi/2$ and $+\pi/2$, and the argument of every power $x^m$ lying between $-m\pi/2$ and $+m\pi/2$.

Suppose next that the real part of $x$ is negative and that the imaginary part is positive. Let us now take as the beginning and end of the paths of integration in (13), (14), (15) a point still at infinity, but whose argument instead of being $\pi$ has some value between $\pi$ and $3\pi/2$, say $5\pi/4$, and take values of the function under the sign of integration which are reconcilable with those formerly taken. These changes of path do not affect the values of the integrals. Proceeding as before, we have now to deal with $(1-u/x)^{-\alpha}$ where the argument of $u$ is now $5\pi/4$ instead of $\pi$; this is of the form $(1-t)^{-\alpha}$ where the argument of $t$ is somewhere between $\pi/4$ and $3\pi/4$ instead of between $\pi/2$ and $3\pi/2$; the integral on the right-hand side of (24) has now the same absolute value as

$$\int_{-\infty}^{0} e^{\frac{v(1+i)}{\sqrt{2}}} \left(v\frac{1+i}{\sqrt{2}\,.\,x}\right)^{\alpha-\rho} R_s \frac{1+i}{\sqrt{2}}\, dv,$$

where $v$ is real and negative and $R_s$ denotes the remainder after $s$ terms of the expansion of $\left(1-\dfrac{v(1+i)}{x\sqrt{2}}\right)^{-\alpha}$. By Art. 2, $R_s$ is certainly less in absolute value than the $(s+1)$th term of the expansion multiplied by $2^{\frac{\beta}{2}}$ where $\beta$ is a positive integer such that $\beta-\alpha$ is positive, and as

$$\int_{-\infty}^{0} e^{\frac{v(1+i)}{\sqrt{2}}}\, v^m dv,$$

when $m+1$ is positive, is less in absolute value than $2^{\frac{m+1}{2}}\,\Pi(m)$, it follows that $R_s'$ is less in absolute value than the $(s+1)$th term of the series in (22) multiplied by

$2^{\frac{\alpha+\beta-\rho+s+1}{2}}$ *. As there is no superior limit to the value of $s$ for which this is true, and as, by taking $x$ great enough, terms at the beginning may be made to outweigh as much as we please any finite number of succeeding terms, it follows that $x$ may be taken so great that for any given value of $s$ the error in taking $s$ terms of the series in brackets in (22) instead of the whole expression will have a modulus less than that of the next term multiplied by $1+\epsilon$, where $\epsilon$ is any assigned small positive quantity.

If both the real and imaginary parts of $x$ are negative the same result follows by taking for the point at infinity in (13), (14), (15), one whose argument lies between $\pi/2$ and $\pi$.

Equation (26) is thus true in the sense indicated when the real part of $x$ is negative, the argument of every power $x^m$ lying between $-m\pi$ and $-m\pi/2$ or between $+m\pi/2$ and $+m\pi$.

I proceed to obtain the relation connecting the divergent series (10) with the two convergent solutions of equation (7). If in (26) we change $\alpha$ into $1-\alpha$, $\rho$ into $2-\rho$, $x$ into $ye^{-\pi i}$, it becomes

$$\Pi(-\alpha)\Pi(\rho-2)F(1-\alpha;\ 2-\rho;\ -y)+\Pi(\rho-\alpha-1)\Pi(-\rho)e^{(1-\rho)\pi i}y^{\rho-1}F(\rho-\alpha;\ \rho;\ -y)$$
$$=\Pi(-\alpha)\Pi(\rho-\alpha-1)e^{(1-\alpha)\pi i}y^{\alpha-1}\phi(1-\alpha,\ \rho-\alpha;\ 1/y),$$

wherein the argument of every power $y^m$ lies between 0 and $2m\pi$. After multiplication by $e^y$, bearing in mind equations (8), (9), and writing $x$ instead of $y$, this may be written in the form

$$\Pi(\rho-\alpha-1)\Pi(-\rho)F(\alpha;\ \rho;\ x)+\Pi(-\alpha)\Pi(\rho-2)e^{(\rho-1)\pi i}x^{1-\rho}F(\alpha-\rho+1;\ 2-\rho;\ x)$$
$$=\Pi(-\alpha)\Pi(\rho-\alpha-1)e^{(\rho-\alpha)\pi i}e^{x}x^{\alpha-\rho}\phi(1-\alpha,\ \rho-\alpha;\ 1/x)\ \ldots\ldots\ldots\ldots(27).$$

This is true in the sense that if the real part of $x$ be negative the error in stopping the series on the right after $s$ terms is less in absolute value than the next term, provided $s+1-\alpha$ and $s+\rho-\alpha$ are positive, and that whether the real part of $x$ be positive or negative $x$ can be taken so great that the error in stopping after $s$ terms is less in absolute value than the next term multiplied by $1+\epsilon$, where $\epsilon$ has any assigned positive value and $s$ has any given value.

The expression on the right of (27) is a multiple of (15) and may be obtained from that integral if $x$ has its real part positive, by changing the starting point to some point at infinity whose real part is negative but which subtends with the origin an obtuse angle at the point $x$, and writing $u=x+v$.

Equation (27) may also be obtained directly by replacing the path of $u$ in (12) by four other portions in a different manner. For instance instead of (12) we may write

$$-\int_c^{(0,\,x,\,0-,\,x-)} e^u(u-x)^{-\alpha}u^{\alpha-\rho}\,du,$$

(Pochhammer: "Ueber ein Integral mit doppeltem Umlauf," *Math. Annal.* Band XXXV.), and if the real part of $x$ is negative and its imaginary part positive divide this path into

* This reasoning, in fact, proves that if the argument of $x$ is $\pi\pm\gamma$, ($\gamma$ acute), the multiplier is $[\operatorname{cosec}(\theta+\gamma)]^{\beta}(\sec\theta)^{\alpha-\rho+s+1}$ provided $\theta$ and $\theta+\gamma$ are acute; also $\alpha$ and $\alpha-\rho+1$ may be interchanged.

four others, that in (13) taken forwards, that in (15) taken forwards, that in (13) taken backwards, and that in (15) taken backwards. If the point $x$ is differently situated a slight modification is necessary.

In the case which occurs in connection with Bessel Functions*, viz., that in which $\rho = 2\alpha$, Hankel (*loc. cit.*) considers the integral

$$\int_0^\infty e^{-xr} r^{n-\frac{1}{2}} (1 + ri)^{n-\frac{1}{2}} dr.$$

Although as Hankel proves, the remainder after $s$ terms of $(1+ri)^{n-\frac{1}{2}}$ is less in absolute value than the next term (provided $n - s - \frac{1}{2}$ is negative), his inference that a similar statement holds for the integrals, is only valid when $x$ is wholly real. For if $u$, $v$, $w$ be complex functions of $r$ which is wholly real, and if mod. $u <$ mod. $v$, we are not justified in inferring that mod. $\int_a^b uwdr <$ mod. $\int_a^b vwdr$.

Gray and Mathews (*Bessel Functions*, p. 69) apply Lagrange's form of the remainder in Taylor's series to the case of a complex function $\left(1 \pm \frac{\xi}{2ix}\right)^{n-\frac{1}{2}}$. Although this is a valid form of a superior limit to the modulus of the remainder, we are not justified in assuming that $\left(1 + \frac{\theta\xi}{2ix}\right)^{n-s-\frac{1}{2}}$ and $\left(1 - \frac{\theta'\xi}{2ix}\right)^{n-s-\frac{1}{2}}$ ($n - s - \frac{1}{2}$ being negative) are both less in absolute value than unity, where $\theta$, $\theta'$, $\xi$ are real, unless $x$ be wholly real.

4. The theorem indicated by equation (26) is a particular case of the following which will be proved by induction:—

$$\frac{\Pi(\alpha_1 - 1)\,\Pi(-\rho_1)\,\Pi(-\rho_2)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_2)\,\Pi(-\alpha_3)\ldots\Pi(-\alpha_n)} F(\alpha_1, \alpha_2, \ldots \alpha_n;\ \rho_1, \rho_2, \ldots \rho_n;\ x)$$

$$+ \frac{\Pi(\alpha_1 - \rho_1)\,\Pi(\rho_1 - 2)\,\Pi(\rho_1 - \rho_2 - 1)\ldots\Pi(\rho_1 - \rho_n - 1)}{\Pi(\rho_1 - \alpha_2 - 1)\,\Pi(\rho_1 - \alpha_3 - 1)\ldots\Pi(\rho_1 - \alpha_n - 1)} x^{1-\rho_1} F(\alpha_1 - \rho_1 + 1, \ldots \alpha_n - \rho_1 + 1;$$
$$2 - \rho_1, \rho_2 - \rho_1 + 1, \ldots \rho_n - \rho_1 + 1;\ x)$$

$+ (n-1)$ other terms analogous to the last

$$= \frac{\Pi(\alpha_1 - 1)\,\Pi(\alpha_1 - \rho_1)\ldots\Pi(\alpha_1 - \rho_n)}{\Pi(\alpha_1 - \alpha_2)\,\Pi(\alpha_1 - \alpha_3)\ldots\Pi(\alpha_1 - \alpha_n)} x^{-\alpha_1} F(\alpha_1, \alpha_1 - \rho_1 + 1, \ldots \alpha_1 - \rho_n + 1;\ \alpha_1 - \alpha_2 + 1, \ldots$$
$$\alpha_1 - \alpha_n + 1;\ -1/x) \ldots\ldots(28),$$

wherein the argument of every power $x^m$ lies between $-m\pi$ and $+m\pi$, the symbol of equality being interpreted in the sense that:—

(A) If the real part of $x$ is positive, the error committed by stopping the series on the right-hand side after $s$ terms is less in absolute value than the next term, provided $s$ exceeds a certain number. Of the series on the left we may select the first so that $1, \rho_1, \ldots \rho_n$, are in ascending order as also $\alpha_2, \alpha_3, \ldots \alpha_n$. If $a$ is any

* The semiconvergent series for $J_n(x)$ may be readily obtained by forming the equation satisfied by $e^x . I_n(x)$ and using the analogues of (26), (27).

fractional number let $[a]$ denote, if $a$ is negative, zero, if $a$ is positive the integer next higher than $a$. Then $s$ is not to be less than the greater of the integers $[\alpha_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$, $[\rho_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

(B) Whether the real part of $x$ is positive or negative*, $x$ can be taken so great that for any assigned value of $s$ the error in stopping after $s$ terms is less in absolute value than the next term multiplied by $1+\epsilon$, where $\epsilon$ is any assigned positive quantity however small.

It is to be noted that in the enunciation of (A) the additional factors occurring in the numerator and denominator of the second term omitted are all positive, and the argument of $x$ is restricted to a range of $\pi$, including that value which makes all the terms omitted real and of alternate signs; *some* restriction on the argument or on the amplitude of $x$ being, as remarked in Art. 1, not merely incidental to our method of proof but essentially necessary from the nature of the theorem.

5. We will first prove by induction that there is one solution of the differential equation satisfied by

$$F(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \rho_2, \dots \rho_n;\ -x)$$

which can be written in the form $Ce^{-x}x^{\sum(\alpha-\rho)}$ where, as $x$ increases indefinitely, having its real part positive, $C$ tends to a fixed limit. (The minus sign has been inserted before the $x$ as it is easier to reason about a negative quantity when we call it $-x$ than when we call it $+x$.) This is true also if the real part of $x$ is negative but not required for the present purpose. Let us assume that this is true when there are $n$ $\alpha$'s and $n$ $\rho$'s, and introduce another $\alpha$ and another $\rho$ denoted simply by $\alpha$ and $\rho$. The differential equation for the new hypergeometric function is satisfied by

$$x^{1-\rho}\int_{\infty}^{(x)} (v-x)^{\rho-\alpha-1} v^{\alpha-1} \phi(v)\, dv,$$

where $\phi(v)$ is the solution of the old series referred to (see Pochhammer). Writing

$$\phi(v) = Ce^{-v}v^{\sum_1^n(\alpha_r-\rho_r)} = Ce^{-v}v^m$$

and making the substitution $v = x + u$, the above solution may be written in the form

$$e^{-x}x^{\alpha-\rho+m}\int_{\infty}^{(0)} Ce^{-u}u^{\rho-\alpha-1}(1+u/x)^{\alpha+m-1}\, du \quad \dots\dots\dots\dots\dots\dots\dots\dots (29).$$

Let us assume in the first instance that $\rho - \alpha$ is positive; this is then a multiple (depending on the unspecified arguments) of the line integral

$$e^{-x}x^{\alpha-\rho+m}\int_0^{\infty} Ce^{-u}u^{\rho-\alpha-1}(1+u/x)^{\alpha+m-1}\, du.$$

If $(1+u/x)^{\alpha+m-1}$ be now expanded in powers of $u/x$ the modulus of the remainder after a certain term will be less than that of the next term, and it is accordingly

* If the latter, it will also be shown that the error is less than a certain multiple of the next term ($s$ restricted as before).

evident that the above may be written in the form $C'e^{-x}x^{\alpha-\rho+m}$ where, by increasing $x$ sufficiently, $C'$ can be made as nearly as we please equal to a certain constant.

Also in case $\rho-\alpha$ be negative, on integrating (29) by parts the integral may be written in the form

$$\frac{1}{\rho-\alpha}\left[Ce^{-u}u^{\rho-\alpha}(1+u/x)^{\alpha+m-1}\right]_{\infty}^{\infty}-\frac{1}{\rho-\alpha}\overline{\int_{\infty}^{(0)}}e^{-u}u^{\rho-\alpha}(1+u/x)^{\alpha+m-2}\left\{\frac{\alpha+m-1}{x}C-C(1+u/x)\right.$$
$$\left.+(1+u/x)\,dC/du\right\}du.$$

The expression in the square brackets vanishes at the infinite limits, and by making $x$ great enough $dC/du$ may be made as small as we please by hypothesis, and the above integral, if $x$ be large enough, can thus be made as nearly as we please equal to

$$-\frac{1}{\rho-\alpha}\overline{\int_{\infty}^{(0)}}e^{-u}u^{\rho-\alpha}(1+u/x)^{\alpha+m-2}\left\{\frac{\alpha+m-1}{x}C-C(1+u/x)\right\}du,$$

from which it is evident that any limits to the value of $\rho-\alpha$ may be extended by unity, and therefore, for all values of $\rho-\alpha$, (29) may be written in the form

$$C'e^{-x}x^{\alpha-\rho+\sum_1^n(\alpha_r-\rho_r)}$$

where $C'$ tends to a fixed limit as $x$ increases indefinitely.

6. The solution which does tend to this form is a multiple of

$$\frac{\Pi(-\rho_1)\Pi(-\rho_2)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\Pi(-\alpha_2)\ldots\Pi(-\alpha_n)}F(\alpha_1,\alpha_2,\ldots\alpha_n;\ \rho_1,\ldots\rho_n;\ -x)$$
$$+\sum_{r=1}^{r=n}\frac{\Pi(\rho_r-2)\Pi(\rho_r-\rho_1-1)\ldots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_1-1)\ldots\Pi(\rho_r-\alpha_n-1)}x^{1-\rho_r}F(\alpha_1-\rho_r+1,\ldots\alpha_n-\rho_r+1;\ 2-\rho_r,\rho_1-\rho_r+1,$$
$$\ldots\rho_n-\rho_r+1;\ -x),$$

wherein the argument of $x^m$ lies between $-m\pi/2$ and $+m\pi/2$. This may be seen by making in the theorem indicated by equation (28) the arguments of $x$ in succession $-\pi$ and $+\pi$ and subtracting the results after having multiplied one of them by $e^{2\pi\alpha i}$. It then appears that the above solution is one which, when in it $x$ is made real and positive and sufficiently great, can be made less than a certain multiple of any specified term of the series

$$x^{-\alpha_1}F(\alpha_1,\ \alpha_1-\rho_1+1,\ldots;\ \alpha_1-\alpha_2+1,\ldots;\ +1/x),$$

and therefore must be the one in question, as from equation (28), assumed to hold for the above functions, no other solution can be of this order for infinite values of $x$ which have their real part positive.

7. We require to evaluate the integral

$$\int_0^{\infty}x^mF(\alpha_1,\ \alpha_2,\ldots\alpha_n;\ \rho_1,\ \rho_2,\ldots\rho_n;\ -x)\,dx\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (30),$$

when intelligible, that is when $m+1$ is positive and $m-\alpha_r+1$ is negative, where $\alpha_r$ is the algebraically least $\alpha$. (When $x$ is very great the hypergeometric series is of order $x^{-\alpha_r}$.)

The hypergeometric function is equal to

$$\frac{\Pi(\rho_n-1)}{\Pi(\alpha_n-1)\,\Pi(\rho_n-\alpha_n-1)}\int_0^x x^{1-\rho_n}(x-v)^{\rho_n-\alpha_n-1}v^{\alpha_n-1}F(\alpha_1,\dots\alpha_{n-1};\rho_1,\dots\rho_{n-1};-v)\,dv,$$

provided $\alpha_n$ and $\rho_n-\alpha_n$ are positive. Substituting this value in (30) and changing the order of integration, which can be shown to be legitimate since $m-\alpha_n+1$ is negative, (30) can be written in the form

$$\frac{\Pi(\rho_n-1)}{\Pi(\alpha_n-1)\,\Pi(\rho_n-\alpha_n-1)}\int_0^\infty v^{\alpha_n-1}F(\alpha_1,\dots\alpha_{n-1};\rho_1,\dots\rho_{n-1};-v)\,dv\int_v^\infty x^{m+1-\rho_n}(x-v)^{\rho_n-\alpha_n-1}dx;$$

but the $x$ integral is equal to

$$\frac{\Pi(\alpha_n-m-2)\,\Pi(\rho_n-\alpha_n-1)}{\Pi(\rho_n-m-2)}\,v^{m+1-\alpha_n},$$

and accordingly (30) is equivalent to

$$\frac{\Pi(\rho_n-1)\,\Pi(\alpha_n-m-2)}{\Pi(\alpha_n-1)\,\Pi(\rho_n-m-2)}\int_0^\infty v^m F(\alpha_1,\dots\alpha_{n-1};\rho_1,\dots\rho_{n-1};-v)\,dv.$$

In the same way provided $\alpha_1$ and $\rho_1-\alpha_1$ are positive

$$\int_0^\infty x^m F(\alpha_1;\rho_1;-x)\,dx=\frac{\Pi(\rho_1-1)\,\Pi(\alpha_1-m-2)}{\Pi(\alpha_1-1)\,\Pi(\rho_1-m-2)}\int_0^\infty x^m e^{-x}\,dx$$
$$=\frac{\Pi(\rho_1-1)\,\Pi(\alpha_1-m-2)}{\Pi(\alpha_1-1)\,\Pi(\rho_1-m-2)}\,\Pi(m).$$

Therefore by induction we finally obtain for (30) the value

$$\Pi(m)\,.\prod_{r=1}^{r=n}\frac{\Pi(\rho_r-1)\,\Pi(\alpha_r-m-2)}{\Pi(\alpha_r-1)\,\Pi(\rho_r-m-2)}\quad\dots\dots\dots\dots\dots\dots\dots\dots(31),$$

provided, in addition to the conditions necessary to make (30) intelligible, all such quantities as $\alpha_r$ and $\rho_r-\alpha_r$ are positive.

These latter conditions may however be removed. Since

$$F(\alpha_1+1,\alpha_2,\dots\alpha_n;\rho_1,\rho_2,\dots\rho_n;-x)-F(\alpha_1,\alpha_2,\dots\alpha_n;\rho_1,\rho_2,\dots\rho_n:-x)$$
$$=\frac{x}{\alpha_1}\frac{d}{dx}F(\alpha_1,\alpha_2,\dots\alpha_n;\rho_1,\rho_2,\dots\rho_n;-x),$$

by multiplying both sides by $x^m$ and integrating we obtain

$$\int_0^\infty x^m F(\alpha_1+1,\alpha_2,\dots\alpha_n;\rho_1,\dots\rho_n;-x)\,dx-\int_0^\infty x^m F(\alpha_1,\dots\alpha_n;\rho_1,\dots\rho_n;-x)\,dx$$
$$=\left[\frac{x^{m+1}}{\alpha_1}F(\alpha_1,\dots\alpha_n;\rho_1,\dots\rho_n;-x)\right]_0^\infty-\frac{m+1}{\alpha_1}\int_0^\infty x^m F(\alpha_1,\dots\alpha_n;\rho_1,\dots\rho_n;-x)\,dx.$$

If the second integral in the left-hand member is intelligible, so also is the first; the expression in square brackets then vanishes at the limits and

$$\int_0^\infty x^m F(\alpha_1,\dots\alpha_n;\rho_1,\dots\rho_n;-x)\,dx=\frac{\alpha_1}{\alpha_1-m-1}\int_0^\infty x^m F(\alpha_1+1,\alpha_2,\dots\alpha_n;\rho_1,\dots\rho_n;-x)\,dx,$$

and accordingly any inferior limit imposed on any $\alpha$ may be extended by unity, the other $\alpha$'s and $\rho$'s being kept unchanged, provided in (30) the integral remains intelligible.

Also from the equation

$$F(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \rho_2, \dots \rho_n;\ -x) - F(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \rho_2, \dots \rho_{n-1},\ \rho_n - 1;\ -x)$$
$$= \frac{-x}{\rho_n - 1}\frac{d}{dx} F(\alpha_1, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ -x) \quad \dots\dots\dots\dots(32),$$

by multiplying both sides by $x^m$ and integrating we obtain

$$\int_0^\infty x^m F(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \rho_2, \dots \rho_n;\ -x)\,dx - \int_0^\infty x^m F(\alpha_1, \dots \alpha_n;\ \rho_1, \rho_2, \dots \rho_{n-1},\ \rho_n - 1;\ -x)\,dx$$
$$= \frac{-1}{\rho_n - 1}\Big[x^{m+1} F(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ -x)\Big]_0^\infty$$
$$+ \frac{m+1}{\rho_n - 1}\int_0^\infty x^m F(\alpha_1, \dots \alpha_n;\ \rho_1, \rho_2, \dots \rho_n;\ -x)\,dx \quad \dots\dots\dots\dots(33).$$

If (30) is intelligible all the terms in this equation are finite and that in square brackets is zero at the limits and thus we obtain

$$\int_0^\infty x^m F(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \rho_2, \dots \rho_{n-1},\ \rho_n - 1;\ -x)\,dx$$
$$= \frac{\rho_n - m - 2}{\rho_n - 1}\int_0^\infty x^m F(\alpha_1, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ -x)\,dx \dots\dots(34),$$

and accordingly any limit to the value of any $\rho$ may be extended by unity, the other $\alpha$'s and $\rho$'s being kept unchanged. The result stated as to the value of the integral (30) has thus been established *.

8. It is to be noted that the proofs given of the results of Arts. 6, 7 for functions of any order assume the truth of equation (28) for functions of the same order. We now proceed, assuming the results of Arts. 6, 7 and equation (28) for functions of any and the same order, to extend equation (28) to functions of the next higher order by the introduction of another $\rho$ and another $\alpha$. We do so in the present Article, taking the equation in sense (A) but subject to the restrictions that each of the quantities $\rho_2 - \alpha_2, \dots \rho_n - \alpha_n,\ \rho - \alpha,\ \alpha_1 + 1 - \rho_2, \dots \alpha_1 + 1 - \rho_n,\ \alpha_1 + 1 - \rho$, is positive. In Art. 9 we will extend the equation in the sense (B), and in Art. 10 the restrictions introduced in the present Article for the sense (A) will be removed.

As indicated by Pochhammer the equation satisfied by $F(\alpha, \alpha_1, \alpha_2, \dots \alpha_n;\ \rho, \rho_1, \dots \rho_n;\ +x)$ is satisfied by

$$x^{1-\rho}\int (v - x)^{\rho-\alpha-1} v^{\alpha-1}\,\phi(v)\,dv \quad \dots\dots\dots\dots(35),$$

where $\phi(v)$ is any solution of that satisfied by $F(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ +v)$ and the path of integration is a closed one such that the final value of $(v - x)^{\rho-\alpha-1} v^{\alpha-1}\phi(v)$ differs by zero from the initial one.

* This result may be generalized by omitting any number of $\alpha$'s, if the integral remains finite; write $x = y/\alpha$ and then make $\alpha$ infinite.

Let the path be one which makes a circuit round the point $x$ in the positive direction, then round the origin in the positive direction, then round the point $x$ in the negative direction and finally round the origin in the negative direction. Suppose in the first instance the real part of $x$ to be positive.

Such a path is equivalent to the paths $ABCA$, $ADBA$, $ACBDA$ which may be replaced by the four portions $ABCA$, $ADBA$, $ACBA$, $ABDA$, (Fig. 3).

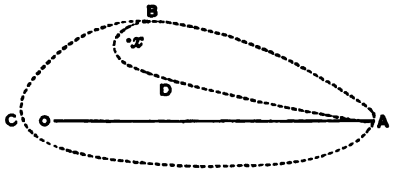



Fig. 3.

Let $A$ be a point $h$ on the axis of real quantities and let $\phi(v)$ be

$$CF(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \rho_2, \dots \rho_n;\ +v)$$

$$+\sum_{r=1}^{r=n} C_r v^{1-\rho_r}\ F(\alpha_1-\rho_r+1, \dots \alpha_n-\rho_r+1;\ 2-\rho_r, \rho_1-\rho_r+1, \dots \rho_n-\rho_r+1;\ +v) \dots\dots\dots(36),$$

those values being taken which make the initial arguments of every power of $v$ zero at the point $h$ (before multiplication by $C$, $C_r$), and make the initial argument of $(v-x)^{\rho-\alpha-1}$ diminish indefinitely as $h$ increases indefinitely.

On examining the values of the arguments at different points it will be seen that the second and fourth portions of the path together contribute to the integral (35) the expression

$$(e^{2\pi i(\alpha-\rho)} - e^{2\pi i\alpha})\, x^{1-\rho} \int_h^{\overline{(x)}} C\,(v-x)^{\rho-\alpha-1}\, v^{\alpha-1}\, F(\alpha_1, \alpha_2, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ +v)\, dv$$

$$+\sum_{r=1}^{r=n} (e^{2\pi i(\alpha-\rho)} - e^{2\pi i(\alpha-\rho_r)})\, x^{1-\rho} \int_h^{\overline{(x)}} C_r\,(v-x)^{\rho-\alpha-1}\, v^{\alpha-\rho_r}\ F(\alpha_1-\rho_r+1, \dots \alpha_n-\rho_r+1;\ 2-\rho_r, \dots$$

$$\dots \rho_n-\rho_r+1;\ +v)\, dv \dots\dots\dots\dots(37),$$

the initial arguments being taken as above.

If the differential coefficient of this expression with respect to $h$ be written down it will be evident that in virtue of equation (28) assumed for the function of the $(n+1)$th order, i.e. that which satisfies the differential equation of the $(n+1)$th order, provided we take

$$\left.\begin{aligned} C\,(e^{2\pi i(\alpha-\rho)} - e^{2\pi\alpha i}) &= \frac{\Pi(\alpha_1-1)\,\Pi(-\rho_1)\dots\Pi(-\rho_n)}{\Pi(-\alpha_2)\,\Pi(-\alpha_3)\dots\Pi(-\alpha_n)} \\ C_r(e^{2\pi i(\alpha-\rho)} - e^{2\pi i(\alpha-\rho_r)}) &= \frac{\Pi(\alpha_1-\rho_r)\,\Pi(\rho_r-2)\,\Pi(\rho_r-\rho_1-1)\dots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_2-1)\,\Pi(\rho_r-\alpha_3-1)\dots\Pi(\rho_r-\alpha_n-1)} \end{aligned}\right\} \dots\dots\dots(38),$$

this differential coefficient when $h$ increases indefinitely will be of the order $h^{\rho-\alpha_1-2}$; and the same is true also for a complex value of $h$ provided the argument of $h$ is kept between $-\pi$ and $+\pi$ and the value of the function to be integrated is reconcilable

with that previously taken. Accordingly provided $\rho - \alpha_1 - 1$ is negative, a condition which we have for the present supposed satisfied, the expression (37) will remain finite when $h$ increases indefinitely, and its value will be unaltered if for $h$ we substitute any infinite limit whose argument lies between $-\pi$ and $+\pi$ and if the value of the function to be integrated is reconcilable with that already taken.

Now let $h$ be increased indefinitely. We have also supposed for the present that $\rho - \alpha$ is positive; each symbol of integration in (37) may therefore be replaced by

$$(e^{2\pi i(\rho-\alpha)} - 1)\int_x^\infty$$

in which the infinite limit is on the production of the line joining the origin to the point $x$.

With the above values of the constants $C$, $C_r$, and assuming the theorem to hold for the functions of the $(n+1)$th order, the expression (37) may be written

$$(e^{2\pi i(\rho-\alpha)} - 1)\frac{\Pi(\alpha_1 - 1)\,\Pi(\alpha_1 - \rho_1)\ldots\Pi(\alpha_1 - \rho_n)}{\Pi(\alpha_1 - \alpha_2)\,\Pi(\alpha_1 - \alpha_3)\ldots\Pi(\alpha_1 - \alpha_n)}\int_x^\infty x^{1-\rho}(v-x)^{\rho-\alpha-1}v^{\alpha-\alpha_1-1}$$

$$F(\alpha_1, \alpha_1 - \rho_1 + 1, \ldots \alpha_1 - \rho_n + 1;\ \alpha_1 - \alpha_2 + 1, \alpha_1 - \alpha_3 + 1, \ldots \alpha_1 - \alpha_n + 1;\ -1/v)\,dv\ldots(39).$$

Expanding the divergent hypergeometric series in descending powers of $v$ and integrating the terms successively we obtain

$$(e^{2\pi i(\rho-\alpha)} - 1)\frac{\Pi(\rho-\alpha-1)\,\Pi(\alpha_1-\rho)\,\Pi(\alpha_1-1)\,\Pi(\alpha_1-\rho_1)\ldots\Pi(\alpha_1-\rho_n)}{\Pi(\alpha_1-\alpha)\,\Pi(\alpha_1-\alpha_2)\ldots\Pi(\alpha_1-\alpha_n)}x^{-\alpha_1}F(\alpha_1, \alpha_1-\rho+1, \ldots$$

$$\alpha_1-\rho_n+1;\ \alpha_1-\alpha+1, \alpha_1-\alpha_2+1, \ldots \alpha_1-\alpha_n+1;\ -1/x)\ldots\ldots(39a),$$

the argument of $x^{-\alpha_1}$ lying between $-\alpha_1\pi/2$ and $+\alpha_1\pi/2$; and by expressing the error as an integral, the theorem being assumed for the function of the $(n+1)$th order, it appears that, provided $\alpha_1 + s$ and all the quantities of the type $\alpha_1 - \rho_r + 1 + s$ are positive, the error in stopping the divergent series at the $s$th term has a modulus less than that of the next term.

Returning to (35), the sum of the portions contributed by the first and third portions of the path is

$$(1 - e^{2\pi i(\alpha-\rho)})\int_h^{\overline{(x,0)}} x^{1-\rho}C(v-x)^{\rho-\alpha-1}v^{\alpha-1}F(\alpha_1, \ldots \alpha_n;\ \rho_1, \ldots \rho_n;\ +v)\,dv$$

$$+\sum_{r=1}^{r=n}(1 - e^{2\pi i(\alpha-\rho)})\int_h^{\overline{(x,0)}} C_r x^{1-\rho}(v-x)^{\rho-\alpha-1}v^{\alpha-\rho_r}$$

$$F(\alpha_1-\rho_r+1, \ldots \alpha_n-\rho_r+1;\ 2-\rho_r, \ldots \rho_n-\rho_r+1;\ +v)\,dv\ldots\ldots(40),$$

the initial values of the arguments at the point $h$ being the same as in (36). The differential coefficient of this with respect to $h$ is of the order $h^{\rho-\alpha_1-2}$ owing to the particular values assigned to $C$, $C_r$, and accordingly (40) like (37) remains finite

when $h$ is increased indefinitely provided $\rho - \alpha_1 - 1$ is negative. We now expand $(v-x)^{\rho-\alpha-1}$ in ascending powers of $x$; the coefficient of $x^{1-\rho+m}$ in (40) is therefore

$$(1-e^{2\pi i(\alpha-\rho)})\frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\Pi(m)}\bar{\int}_h^{(0)}\Big[Cv^{\rho-m-2}F(\alpha_1,\dots\alpha_n;\ \rho_1,\dots\rho_n;\ +v)$$

$$+\sum_{r=1}^{r=n}C_r v^{\rho-m-1-\rho_r}\ F(\alpha_1-\rho_r+1,\dots\alpha_n-\rho_r+1;\ 2-\rho_r,\dots\rho_n-\rho_r+1;\ +v)\Big]\,dv\ \dots\dots(41),$$

wherein all the powers of $v$ are initially real before multiplication by the complex coefficients $C$, $C_r$. The successive values of $m$ are 0, 1, 2, &c.

We proceed to evaluate the integral in this when $h$ is made infinite. By considering that, owing to the particular values assigned to $C$, $C_r$, its differential coefficient with respect to $h$ is of the order $h^{\rho-m-\alpha_1-2}$, it appears that the integral remains finite when $h$ increases indefinitely, (provided $\rho - m - \alpha_1 - 1$ is negative, which is certainly true if

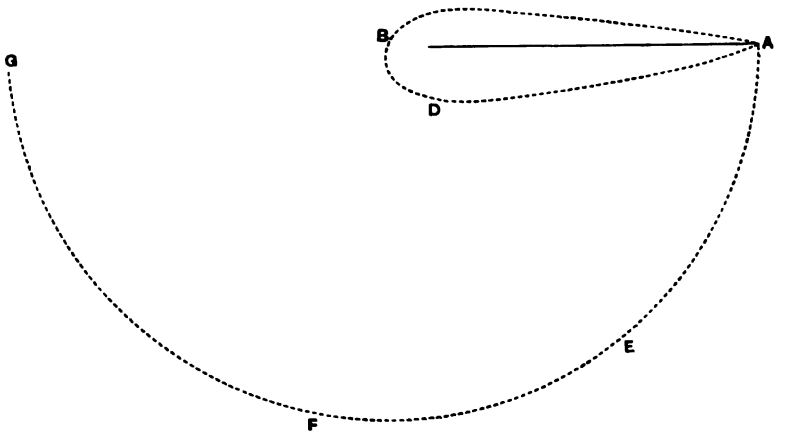



Fig. 4.

as already supposed $\rho - \alpha_1 - 1$ is negative), and as in the case of (37) this is true also for a complex value of $h$ provided the argument of $h$ is kept between $-\pi$ and $+\pi$ and the value of the function to be integrated is reconcilable with that previously taken. Accordingly by changing the argument of $h$ to $-\pi$ as in the path $GFEABDAEFG$, (Fig. 4), the symbol of integration in (41) may be changed into $\int_{-\infty}^{(0)}$ where all the powers of $v$ in the function to be integrated have zero argument (before multiplication by the complex coefficients $C$, $C_r$) at the point in which the path intersects the positive part of the axis of real quantities, and the initial and final limits of integration are not merely both negative and infinite but the same. (By changing the argument of $h$ to $+\pi$ instead, we might obtain the symbol $\bar{\int}_{-\infty}^{(0)}$ with different values for the arguments of the functions to be integrated; the evaluation of the integral would

however lead to the same result as that to be presently deduced for the present case, as of course it should.)

Let us first suppose that $\rho - m - 1$ and all the quantities $\rho - m - \rho_r$ are positive; the integral can then be expressed as a line integral, and attending to the values of the arguments and of $C$, $C_r$, it is in fact

$$\int_0^\infty \Bigg[ e^{(\rho-2a)\pi i} \frac{\sin(\rho-m)\pi}{\sin\rho\pi} \cdot \frac{\Pi(\alpha_1-1)\Pi(-\rho_1)\dots\Pi(-\rho_n)}{\Pi(-\alpha_2)\Pi(-\alpha_3)\dots\Pi(-\alpha_n)} u^{\rho-m-2} F(\alpha_1,\dots\alpha_n;\ \rho_1,\dots\rho_n;\ -u)$$

$$+\sum_{r=1}^{r=n} e^{(\rho+\rho_r-2a)\pi i} \frac{\sin(\rho-m-\rho_r)\pi}{\sin(\rho_r-\rho)\pi} \cdot \frac{\Pi(\alpha_1-\rho_r)\Pi(\rho_r-2)\Pi(\rho_r-\rho_1-1)\dots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_2-1)\dots\Pi(\rho_r-\alpha_n-1)} u^{\rho-m-1-\rho_r}$$

$$F(\alpha_1-\rho_r+1,\dots\alpha_n-\rho_r+1;\ 2-\rho_r,\ \rho_1-\rho_r+1,\ \dots\rho_n-\rho_r+1;\ -u)\Bigg] du.$$

Using the values given in (31) for each of the $n+1$ terms of this integral and making use of the relation $\Pi(n-1)\Pi(-n) = \pi \operatorname{cosec} n\pi$, the above may be written

$$\pi^2 \frac{\Pi(\alpha_1-\rho+m)\Pi(\alpha_2-\rho+m)\dots\Pi(\alpha_n-\rho+m)}{\Pi(\rho_1-\rho+m)\Pi(\rho_2-\rho+m)\dots\Pi(\rho_n-\rho+m)\Pi(1-\rho+m)}$$

$$\Bigg[ -e^{(\rho-2a)\pi i} \frac{\sin\alpha_2\pi\sin\alpha_3\pi\dots\sin\alpha_n\pi}{\sin\rho\pi\sin\rho_1\pi\dots\sin\rho_n\pi}$$

$$+\sum_{r=1}^{r=n} e^{(\rho+\rho_r-2a)\pi i} \frac{\sin(\alpha_2-\rho_r)\pi\sin(\alpha_3-\rho_r)\pi\dots\sin(\alpha_n-\rho_r)\pi}{\sin\rho_r\pi\sin(\rho-\rho_r)\pi\sin(\rho_1-\rho_r)\pi\dots\sin(\rho_n-\rho_r)\pi}\Bigg] \quad\dots\dots\dots (42).$$

But the expression in square brackets is equivalent to

$$e^{(\rho-2a)\pi i} \cdot e^{\rho\pi i} \frac{\sin(\rho-\alpha_2)\pi\sin(\rho-\alpha_3)\pi\dots\sin(\rho-\alpha_n)\pi}{\sin\rho\pi\sin(\rho-\rho_1)\pi\sin(\rho-\rho_2)\pi\dots\sin(\rho-\rho_n)\pi},$$

for it is easily seen that this last may be written in the form

$$e^{(\rho-2a)\pi i}\left\{\frac{A}{\sin\rho\pi} + \frac{A_1}{\sin(\rho-\rho_1)\pi} + \dots\right\},$$

where $A$, $A_1$, etc. are quantities independent of $\rho$ and their evaluation in the usual way leads to the result stated.

The value of (41) is then

$$(e^{2\pi i(\rho-\alpha)}-1)\pi^2 \frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\Pi(m)} \cdot \frac{\Pi(\alpha_1-\rho+m)\Pi(\alpha_2-\rho+m)\dots\Pi(\alpha_n-\rho+m)}{\Pi(\rho_1-\rho+m)\Pi(\rho_2-\rho+m)\dots\Pi(\rho_n-\rho+m)\Pi(1-\rho+m)}$$

$$\times \frac{\sin(\rho-\alpha_2)\pi\sin(\rho-\alpha_3)\pi\dots\sin(\rho-\alpha_n)\pi}{\sin\rho\pi\sin(\rho-\rho_1)\pi\dots\sin(\rho-\rho_n)\pi} \quad\dots\dots\dots\dots\dots\dots\dots (43).$$

The limit of the expression (41) when $h$ is infinite still has the value (43) even if the additional conditions introduced in the process of evaluation (viz. that $\rho - m - 1$ and all the quantities such as $\rho - m - \rho_r$ should be positive) are not satisfied. For bearing in mind that

$$\frac{d}{dv} F(\alpha_1,\ \alpha_2,\dots\alpha_n;\ \rho_1,\ \rho_2,\dots\rho_n;\ v) = \frac{\alpha_1\alpha_2\dots\alpha_n}{\rho_1\rho_2\dots\rho_n} F(\alpha_1+1,\ \alpha_2+1,\dots\alpha_n+1;\ \rho_1+1,\ \dots\rho_n+1;\ v),$$

and that

$$\frac{d}{dv}\{v^{1-\rho_1}F(\alpha_1-\rho_1+1,\dots\alpha_n-\rho_1+1;\ 2-\rho_1,\ \rho_2-\rho_1+1,\dots\rho_n-\rho_1+1;\ v)\}$$
$$=(1-\rho_1)\,v^{-\rho_1}F(\alpha_1-\rho_1+1,\dots\alpha_n-\rho_1+1;\ 1-\rho_1,\ \rho_2-\rho_1+1,\dots\rho_n-\rho_1+1;\ v),$$

if we write the integral in (41) in the form

$$\overline{\int}_h^{(0)} v^{\rho-m-2}\,\phi(v)\,dv$$

and integrate by parts, we obtain

$$\frac{1}{\rho-m-1}\Big[v^{\rho-m-1}\phi(v)\Big]_h^h-\frac{1}{\rho-m-1}\overline{\int}_h^{(0)} v^{\rho-m-1}\phi'(v)\,dv.$$

But, attending to the values of $C$, $C_r$, given by (38), it appears that the difference between the two values of the expression in square brackets when $v$ is equal to $h$ is zero when $h$ is infinite and that $-\phi'(v)$ only differs from $\phi(v)$ by having all the constants $\alpha_1,\dots\alpha_n$, $\rho_1,\dots\rho_n$ increased by unity, and accordingly we can increase all the quantities $\alpha$, $\alpha_1,\dots\alpha_n$, $\rho$, $\rho_1,\dots\rho_n$ in the equation:—$\underset{h=\infty}{\text{Lt.}}$ (41) = (43), keeping $m$ unaltered.

In a similar manner we can show, by writing the integral in (41) in the form

$$\overline{\int}_h^{(0)} v^{\rho-m-1-\rho_r}\,\psi(v)\,dv.$$

that we can increase $\rho_r$ by unity, keeping the quantities $\alpha$, $\alpha_1,\dots\alpha_n$, $\rho$, $\rho_1,\dots\rho_{r-1}$, $\rho_{r+1},\dots\rho_n$, $m$ unaltered, and still have the equation:—$\underset{h=\infty}{\text{Lt.}}$ (41) = (43).

The initial value of $m$ is zero and in that case (43) reduces to

$$-(e^{2\pi i(\rho-\alpha)}-1)\frac{\Pi(\rho-2)\,\Pi(\rho-\rho_1-1)\,\Pi(\rho-\rho_2-1)\dots\Pi(\rho-\rho_n-1)\,\Pi(\alpha_1-\rho)}{\Pi(\rho-\alpha_2-1)\,\Pi(\rho-\alpha_3-1)\dots\Pi(\rho-\alpha_n-1)},$$

and accordingly the expression (40) contributed by the first and third portions of the path is this multiple of

$$x^{1-\rho}F(\alpha-\rho+1,\ \alpha_1-\rho+1,\ \alpha_2-\rho+1,\dots\alpha_n-\rho+1;$$
$$2-\rho,\ \rho_1-\rho+1,\dots\rho_n-\rho+1;\ +x)\dots\dots\dots\dots(44),$$

wherein the argument of $x^{1-\rho}$ lies between $-\frac{\pi}{2}(1-\rho)$ and $+\frac{\pi}{2}(1-\rho)$.

Again returning to (35), (36),

$$\overline{\int}_c^{(x,\,0,\,x-,\,0-)}(v-x)^{\rho-\alpha-1}v^{\alpha-1}F(\alpha_1,\ \alpha_2,\dots\alpha_n;\ \rho_1,\ \rho_2,\dots\rho_n;\ v)\,dv,$$

wherein the arguments have values reconcilable with those chosen, may be shown to be

$$-4e^{\pi i(\alpha-1)}\sin(\rho-\alpha-1)\pi\sin(\alpha-1)\pi\,\frac{\Pi(\rho-\alpha-1)\,\Pi(\alpha-1)}{\Pi(\rho-1)}\,x^{\rho-1}$$
$$F(\alpha,\ \alpha_1,\dots\alpha_n;\ \rho,\ \rho_1,\dots\rho_n;\ x)\dots\dots\dots\dots(45),$$

wherein the argument of $x^{\rho-1}$ lies between $-(\rho-1)\frac{\pi}{2}$ and $+(\rho-1)\frac{\pi}{2}$, and making use of the values of $C$, $C_r$, given by (38), the integral (35) may after some reduction be written in the form

$$(e^{2\pi i(\rho-\alpha)}-1)\Bigg[\frac{\Pi(\rho-\alpha-1)\,\Pi(\alpha_1-1)\,\Pi(-\rho)\,\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha)\,\Pi(-\alpha_2)\,\Pi(-\alpha_3)\ldots\Pi(-\alpha_n)}F(\alpha,\ \alpha_1,\ldots\alpha_n;\ \rho,\ \rho_1,\ldots\rho_n;\ x)$$

$$+\sum_{r=1}^{r=\infty}\frac{\Pi(\rho-\alpha-1)\,\Pi(\alpha_1-\rho_r)\,\Pi(\rho_r-2)\,\Pi(\rho_r-\rho-1)\,\Pi(\rho_r-\rho_1-1)\ldots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha-1)\,\Pi(\rho_r-\alpha_2-1)\ldots\Pi(\rho_r-\alpha_n-1)}x^{1-\rho_r}F(\alpha-\rho_r+1,\ldots$$

$$\alpha_n-\rho_r+1;\ 2-\rho_r,\ \rho_1-\rho_r+1,\ldots\rho_n-\rho_r+1;\ x)\Bigg]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

Equating (46) to the sum of (44) and (39$a$), and dividing by $(e^{2\pi i(\rho-\alpha)}-1)\,\Pi(\rho-\alpha-1)$, we obtain an equation of the same form as (28) and to be interpreted in the sense (A), but with an additional $\rho$ and an additional $\alpha$.

I do not see how to remove the restrictions imposed at the beginning of this Article without first showing that the theorem is true in the sense (B).

9. We now proceed in a different manner to extend the theorem in the sense (B).

The differential equation for $F(\alpha, \alpha_1, \ldots \alpha_n;\ \rho, \rho_1, \ldots \rho_n;\ +x)$ is satisfied by

$$\int_c^{(-x,\,0,\,-x-,\,0-)} x^{1-\rho}(v+x)^{\rho-\alpha-1}\,v^{\alpha-1}\,\phi(-v)\,dv\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

where $\phi(-v)$ is any solution of that satisfied by

$$F(\alpha_1, \ldots \alpha_n;\ \rho_1, \ldots \rho_n;\ -v)\,dv.$$

(It would be more consistent with what has gone before to change the sign of $v$ in the above. The introduction of the minus sign has however the advantage that the function of $-v$ with which we will be concerned does not involve $i$ explicitly to so great an extent.)

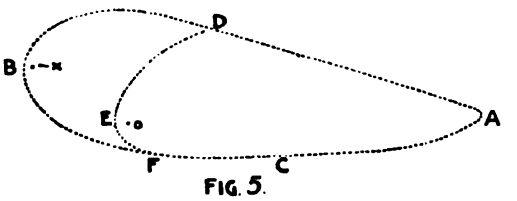


FIG. 5.

The above path is equivalent to the four paths $ABCA$, $ADEFA$, $ACBA$, $AFEDA$ (Fig. 5), $A$ denoting a point $h$ at a great distance on the positive part of the axis of real quantities.

It is assumed in the first instance that the real part of $x$ is positive and the argument of $x^{1-\rho}$ is taken to be between $-(1-\rho)\pi/2$ and $+(1-\rho)\pi/2$. Let $\phi(-v)$ be

$$AF(\alpha_1, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ -v)$$
$$+\sum_{r=1}^{r=n} A_r v^{1-\rho_r} F(\alpha_1-\rho_r+1, \dots \alpha_n-\rho_r+1;\ 2-\rho_r,\ \rho_1-\rho_r+1, \dots \rho_n-\rho_r+1;\ -v),$$

those values being taken which make the initial arguments of every power of $v$ zero at $A$ (before multiplication by $A$, $A_r$) and make the initial argument of $(v+x)^{\rho-\alpha-1}$ diminish indefinitely as $h$ increases indefinitely.

On examining the values of the arguments at different points it will be seen that the first and third portions of the path contribute to the integral (47)

$$(1-e^{2\pi\alpha i})\, x^{1-\rho} \int_h^{\overline{(-x,0)}} A\,(v+x)^{\rho-\alpha-1} v^{\alpha-1} F(\alpha_1, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ -v)\, dv$$
$$+\sum_{r=1}^{r=n} (1-e^{2\pi i(\alpha-\rho_r)})\, x^{1-\rho} \int_h^{\overline{(-x,0)}} A_r\,(v+x)^{\rho-\alpha-1} v^{\alpha-\rho_r} F(\alpha_1-\rho_r+1, \dots;\ 2-\rho_r, \dots \rho_n-\rho_r+1;\ -v)\, dv \dots (48),$$

the initial arguments being taken as above; and assuming the results stated to be true for the function of the $(n+1)$th order, if we take the differential coefficient of this with respect to $h$ it will, by Art. 6, as $h$ increases indefinitely, become of the order of a product of $e^{-h}$ by a certain power of $h$ and therefore diminish indefinitely, provided

$$\left.\begin{aligned} &[(1-e^{2\pi\alpha i})(e^{2\pi\rho i}-1)\,A =]\, 4e^{\pi(\alpha+\rho)i} \sin \alpha\pi \sin \rho\pi \,.\, A = \frac{\Pi(-\rho_1)\dots\Pi(-\rho_n)}{\Pi(-\alpha_1)\dots\Pi(-\alpha_n)} \\ &[(1-e^{2\pi i(\alpha-\rho_r)})(e^{2\pi i(\rho-\rho_r)}-1)\,A_r =]\, 4e^{\pi(\alpha+\rho-2\rho_r)i} \sin(\alpha-\rho_r)\pi \sin(\rho-\rho_r)\pi \,.\, A_r \\ &\qquad = \frac{\Pi(\rho_r-2)\,\Pi(\rho_r-\rho_1-1)\dots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_1-1)\dots\Pi(\rho_r-\alpha_n-1)} \end{aligned}\right\} \dots\dots\dots (49),$$

and therefore with these values this integral will remain finite when $h$ increases indefinitely.

In the expression (48) we now expand $(v+x)^{\rho-\alpha-1}$ in ascending powers of $x$; the coefficient of $x^{1-\rho+m}$ is

$$(-)^m \frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\,\Pi(m)} \Bigg[(1-e^{2\pi\alpha i})\, A \int_h^{\overline{(-x,0)}} v^{\rho-m-2} F(\alpha_1, \dots \alpha_n;\ \rho_1, \dots \rho_n;\ -v)\, dv$$
$$+\sum_{r=1}^{r=n} (1-e^{2\pi i(\alpha-\rho_r)})\, A_r \int_h^{\overline{(-x,0)}} v^{\rho-m-1-\rho_r} F(\alpha_1-\rho_r+1, \dots;\ 2-\rho_r, \dots \rho_n-\rho_r+1;\ -v)\, dv\Bigg] \dots (50),$$

wherein all the terms are initially real, and the successive values of $m$ are 0, 1, 2, &c.

This too remains finite if $h$ increases indefinitely for the same reason as (48).

We will first suppose that all the quantities $\rho-m-1$, $\rho-m-\rho_1, \dots\dots \rho-m-\rho_n$, are positive; the expression in square brackets can then be expressed as a line integral

taken between the limits 0 and $h$, and when $h$ increases indefinitely, using the value given by (31) for the integral (30), it is in fact

$$4e^{\pi(\alpha+\rho)i}\sin\alpha\pi\sin\rho\pi\,.\,A\,\Pi(\rho-m-2)\prod_{s=1}^{s=n}\frac{\Pi(\rho_s-1)\,\Pi(\alpha_s-\rho+m)}{\Pi(\alpha_s-1)\,\Pi(\rho_s-\rho+m)}$$

$$+\sum_{r=1}^{r=n}4e^{\pi i(\alpha+\rho-2\rho_r)}\sin(\alpha-\rho_r)\pi\sin(\rho-\rho_r)\pi\,.\,A_r\Pi(\rho-m-1-\rho_r)\prod_{s=1}^{s=n}\frac{\Pi(\rho_s-\rho_r)\,\Pi(\alpha_s-\rho+m)^*}{\Pi(\alpha_s-\rho_r)\,\Pi(\rho_s-\rho+m)}.$$

Attending to the values of $A$, $A_r$ and making use of the relation

$$\Pi(n-1)\,\Pi(-n)=\Pi\operatorname{cosec}n\pi,$$

this may be written in the form

$$\frac{\pi}{\Pi(m-\rho+1)}\prod_{s=1}^{s=n}\frac{\Pi(\alpha_s-\rho+m)}{\Pi(\rho_s-\rho+m)}\left\{\frac{\sin\alpha_1\pi\sin\alpha_2\pi\ldots\sin\alpha_n\pi}{\sin\rho_1\pi\sin\rho_2\pi\ldots\sin\rho_n\pi\sin(m-\rho+2)\pi}\right.$$

$$\left.+\sum_{r=1}^{r=n}\frac{\sin(\alpha_1-\rho_r+1)\pi\sin(\alpha_2-\rho_r+1)\pi\ldots\sin(\alpha_n-\rho_r+1)\pi}{\sin(\rho_r-1)\pi\sin(\rho_1-\rho_r+1)\pi\ldots\sin(\rho_n-\rho_r+1)\pi\sin(m+1+\rho_r-\rho)\pi}\right\}.$$

But the expression in brackets is equal to

$$(-)^n\frac{\sin(\alpha_1+m-\rho+2)\pi\sin(\alpha_2+m-\rho+2)\pi\ldots\sin(\alpha_n+m-\rho+2)\pi}{\sin(m-\rho+2)\pi\sin(m+1+\rho_1-\rho)\pi\ldots\sin(m+1+\rho_n-\rho)\pi},$$

for it is readily seen that this last can be written in the form

$$\frac{B}{\sin(m-\rho+2)\pi}+\sum_{r=1}^{r=n}\frac{B_r}{\sin(m+1+\rho_r-\rho)\pi},$$

where $B$, $B_r$ are quantities independent of $\rho$ and their evaluation in the usual way leads to the result stated.

Accordingly (50) reduces to

$$(-)^m\frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\Pi(m)}\,\frac{\Pi(\rho-m-2)\,\Pi(\rho-\rho_1-m-1)\ldots\Pi(\rho-\rho_n-m-1)}{\Pi(\rho-\alpha_1-m-1)\,\Pi(\rho-\alpha_2-m-1)\ldots\Pi(\rho-\alpha_n-m-1)}.$$

This result may then be extended to cases in which the conditions that $\rho-m-1$, $\rho-m-\rho_1$, ..... $\rho-m-\rho_n$ should be positive are not satisfied, as is done in a parallel case in Art. 7.

Therefore the portion contributed by the first and third portions of the path is, when $h$ is made infinite,

$$\frac{\Pi(\rho-2)\,\Pi(\rho-\rho_1-1)\ldots\Pi(\rho-\rho_n-1)}{\Pi(\rho-\alpha_1-1)\,\Pi(\rho-\alpha_2-1)\ldots\Pi(\rho-\alpha_n-1)}x^{1-\rho}F(\alpha-\rho+1,\ \alpha_1-\rho+1,\ldots\alpha_n-\rho+1;$$
$$2-\rho,\ \rho_1-\rho+1,\ \rho_2-\rho+1,\ldots\rho_n-\rho+1;\ +x)\ldots\ldots\ldots\ldots(50\,a).$$

* For $s=r$, $\rho_s$ is to be replaced by unity.

Again, the second and fourth portions of the path contribute to the integral (47),

$$(e^{2\pi\rho i}-1)\,x^{1-\rho}\int_h^{(0)} A\,(v+x)^{\rho-\alpha-1}v^{\alpha-1}F(\alpha_1,\dots\alpha_n;\ \rho_1,\rho_2,\dots\rho_n;\ -v)\,dv$$

$$+\sum_{r=1}^{r=n}(e^{2\pi i(\rho-\rho_r)}-1)\,x^{1-\rho}\int_h^{(0)} A_r\,(v+x)^{\rho-\alpha-1}v^{\alpha-\rho_r}F(\alpha_1-\rho_r+1,\dots\alpha_n-\rho_r+1;\ 2-\rho_r,\dots$$

$$\rho_n-\rho_r+1;\ -v)\,dv\ \dots(51).$$

We now expand $(v+x)^{\rho-\alpha-1}$ in descending powers of $x$; the coefficient of $x^{-\alpha-m}$ in the above is

$$(-)^m\frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\,\Pi(m)}\left[(e^{2\pi\rho i}-1)\,A\int_h^{(0)} v^{m+\alpha-1}F(\alpha_1\dots\alpha_n;\ \rho_1\dots\rho_n;\ -v)\,dv\right.$$

$$\left.+\sum_{r=1}^{r=n}(e^{2\pi i(\rho-\rho_r)}-1)\,A_r\int_h^{(0)} v^{m+\alpha-\rho_r}F(\alpha_1-\rho_r+1\dots\alpha_n-\rho_r+1;\ 2-\rho_r\dots\rho_n-\rho_r+1;\ -v)\,dv\right]\dots(52).$$

Suppose at first that the quantities $m+\alpha$, $m+\alpha-\rho_1+1$, ..... $m+\alpha-\rho_n+1$ are all positive; this then reduces to a line integral and the value of the expression in square brackets may be obtained from the value obtained for the expression in square brackets in (50) by changing $m$ into $-m-1$, interchanging $\alpha$ and $\rho$ and then multiplying by $-1$; accordingly the value of (52) is

$$(-)^{m+1}\frac{\Pi(\alpha-\rho+m)}{\Pi(\alpha-\rho)\Pi(m)}\cdot\frac{\Pi(m+\alpha-1)\,\Pi(m+\alpha-\rho_1)\dots\Pi(m+\alpha-\rho_n)}{\Pi(m+\alpha-\alpha_1)\,\Pi(m+\alpha-\alpha_2)\dots\Pi(m+\alpha-\alpha_n)}.$$

This value may as before be extended to the case in which the conditions that all the quantities $m+\alpha$, $m+\alpha-\rho_1+1$, ..... $m+\alpha-\rho_n+1$, must be positive, do not hold. The successive values of $m$ are 0, 1, 2 ... and accordingly we obtain for the part contributed to the integral (47) by the second and fourth portions of the path, the divergent series

$$-\frac{\Pi(\alpha-1)\,\Pi(\alpha-\rho_1)\,\Pi(\alpha-\rho_2)\dots\Pi(\alpha-\rho_n)}{\Pi(\alpha-\alpha_1)\,\Pi(\alpha-\alpha_2)\dots\Pi(\alpha-\alpha_n)}\,x^{-\alpha}F(\alpha,\alpha-\rho+1,\alpha-\rho_1+1,\dots\alpha-\rho_n+1;$$

$$\alpha-\alpha_1+1,\alpha-\alpha_2+1,\dots\alpha-\alpha_n+1;\ -1/x)\dots(53).$$

As regards the remainder in this series after $s$ terms, the remainder after $s$ terms in the expansion of $(v+x)^{\rho-\alpha-1}$ has the origin of $v$ for a multiple point of order $s$, and has, by Art. 2, a modulus less than that of the next term provided $\alpha-\rho+1+s$ is positive; and accordingly bearing in mind the order of $\phi(-v)$ in the neighbourhood of the origin of $v$, the remainder in (53) may be written in the form of a line integral

$$Cx^{-\alpha-s}\int_0^\infty pv^{\alpha-1+s}\phi(-v)\,dv,$$

provided, in addition, all the quantities $\alpha+s$, $\alpha-\rho+1+s$, $\alpha-\rho_1+1+s$, ..... $\alpha-\rho_n+1+s$, are positive, $C$ denoting the numerical factor, and $p$ a quantity whose modulus is less than unity. We are not however justified in assuming that this integral would be increased numerically by replacing $p$ by unity, and hence that the remainder in (53) is less in

absolute value than the first term omitted; for it seems possible that $\phi(-v)$ may change sign between zero and infinity which would invalidate such reasoning*; (if this objection could be removed, this proof would establish the theorem in the sense (A) also). We will show however that if the inferior limit to $s$ imposed by the conditions just laid down be raised by unity, the modulus of the remainder after $s$ terms is less than that of a certain multiple of the next term. Denoting $\sum_1^n (\alpha_r - \rho_r)$ by $\sigma$, as $v$ increases indefinitely $\phi(-v)$ becomes of order $e^{-v}v^{\sigma}$. Suppose that of values of $v$ lying between zero and infinity $v_0$ is that which gives to $v^{\alpha-1+s}\phi(-v)$, which owing to the inferior limit of $s$ being raised is now zero when $v$ is zero, its numerically greatest value. Choosing any positive quantity $\gamma$ less than unity, find a value $v_1$ of $v$ so great that for all greater values $\phi(-v)$ lies between the limits $C'(1 \pm \gamma)\, e^{-v}v^{\sigma}$, $C'$ being a constant which we could find if desired. The integral in the above remainder is therefore less than

$$v_1\phi(-v_0) + C'(1+\gamma)\int_{v_1}^{\infty} e^{-v}v^{\alpha-1+s+\sigma},$$

and therefore less than

$$v_1\,\phi(-v_0) + C'(1+\gamma)\,\Pi(\alpha - 1 + \sigma + s).$$

Therefore the remainder is less in absolute value than a certain multiple, independent of $x$, of the first term omitted.

If the real part of $x$ is negative, the same may be proved, for a different multiple, depending on the argument but not on the modulus of $x$, in a manner similar to that in which the parallel case for the function $F(\alpha;\, \rho;\, x)$ was treated.

Since by taking $x$ great enough, terms at the beginning may be made to outweigh as much as we please any finite number of those that come after, and since there is in the above no superior limit to $s$, it is evident that $x$ may be taken so great that the error committed by stopping the series at any assigned term is less in absolute value than the next term multiplied by $1+\epsilon$, where $\epsilon$ is any assigned positive quantity.

Returning to (47)

$$\overline{\int_c}^{(-x,0,-x-,0-)} x^{1-\rho}(v+x)^{\rho-\alpha-1}v^{\alpha-1}F(\alpha_1, \dots \alpha_n;\, \rho_1, \dots \rho_n;\, -v)\, dv$$

with the values of the arguments reconcilable with those already chosen may be shown to be

$$-4e^{(\rho+\alpha)\pi i}\sin(\rho-\alpha)\,\pi\,\sin\alpha\pi\,.\,\frac{\Pi(\alpha-1)\,\Pi(\rho-\alpha-1)}{\Pi(\rho-1)}\,F(\alpha, \alpha_1, \dots \alpha_n;\, \rho, \rho_1, \dots \rho_n;\, +x),$$

and

$$\overline{\int_c}^{(-x,0,-x-,0-)} x^{1-\rho}(v+x)^{\rho-\alpha-1}v^{\alpha-\rho_r}F(\alpha_1-\rho_r+1, \dots;\, 2-\rho_r, \rho_1-\rho_r+1, \dots;\, -v)\, dv$$

* As $\phi(-v)$ is an integral of the form of that discussed in Art. 5 it may be seen that it does not change sign for values of $v$ between zero and $+\infty$ provided for all values of $r$ from 2 to $n$, $\rho_r - \alpha_r$ is positive. The theorem might therefore be proved in this manner subject to these restrictions which may be removed as in Art. 10. Art. 8 (part of which had gone to press before this was noted) is therefore to a great extent unnecessary.

to be

$$-4e^{(\rho+\alpha-2\rho_r)\pi i}\sin(\rho-\alpha)\pi\sin(\alpha-\rho_r+1)\pi\,.\,\frac{\Pi(\alpha-\rho_r)\,\Pi(\rho-\alpha-1)}{\Pi(\rho-\rho_r-1)}\,x^{1-\rho_r}F(\alpha-\rho_r+1,$$

$$\alpha_1-\rho_r+1,\ldots\alpha_n-\rho_r+1\,;\,2-\rho_r,\rho-\rho_r+1,\ldots\rho_n-\rho_r+1\,;+x)$$

wherein the argument of $x^{1-\rho_r}$ lies between $-\pi(1-\rho_r)$ and $+\pi(1-\rho_r)$, and making use of the values of $A$, $A_r$, given by (49), the integral becomes

$$-\frac{1}{\Pi(\alpha-\rho)}\,\frac{\Pi(\alpha-1)\,\Pi(-\rho)\,\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\,\Pi(-\alpha_2)\ldots\Pi(-\alpha_n)}\,F(\alpha,\alpha_1,\ldots\alpha_n\,;\,\rho,\rho_1,\ldots\rho_n\,;x)$$

$$-\frac{1}{\Pi(\alpha-\rho)}\sum_1^n\frac{\Pi(\alpha-\rho_r)\,\Pi(\rho_r-2)\,\Pi(\rho_r-\rho-1)\,\Pi(\rho_r-\rho_1-1)\ldots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_1-1)\,\Pi(\rho_r-\alpha_2-1)\ldots\Pi(\rho_r-\alpha_n-1)}$$

$$x^{1-\rho_r}F(\alpha-\rho_r+1,\alpha_1-\rho_r+1,\ldots\alpha_n-\rho_r+1\,;\,2-\rho_r,\rho-\rho_r+1,\ldots\rho_n-\rho_r+1\,;+x)\ldots(54).$$

Equating this to the sum of (50$a$) and (53) and then multiplying by $\Pi(\alpha-\rho)$ we obtain a result similar to that indicated by equation (28) in the sense (B). It differs however from that obtained in the sense (A), in having $\alpha$ and $\alpha_1$, $\rho$ and $\rho_1$ interchanged.

10. Before proceeding to remove from equation (28) taken in the sense (A) the restrictions imposed in Art. 8 that certain quantities must be positive, we will first show that if $\alpha_1-\alpha_n$ is positive and if the theorem holds for the remainder after $s$ terms of the function involving $\alpha_1$, $\alpha_n$, it holds also for the remainder after $(s+1)$ terms of the function involving $\alpha_1$, $\alpha_n+1$, the other $\alpha$'s and $\rho$'s being unchanged.

If $\psi(\alpha_1,\ \alpha_2,\ldots\alpha_n;\ \rho_1,\ \rho_2,\ldots\rho_n;\ x)$ denote either any one of the $n+1$ series of the left-hand member of (28) including the constant multiplier, or the sum of the terms at the beginning of the divergent series on the right (including the constant multiplier and the factor $x^{-\alpha_1}$) up to and including the term involving a specified power of $x$, it is easily verified that

$$\psi(\alpha_1,\ \alpha_2,\ldots\alpha_{n-1},\ \alpha_n+1\,;\ \rho_1,\ \rho_2,\ldots\rho_n;\ x)$$

$$=(\alpha_1-\alpha_n)\,\psi(\alpha_1,\ \alpha_2,\ldots\alpha_{n-1},\ \alpha_n;\ \rho_1,\ \rho_2,\ldots\rho_n;\ x)$$

$$-\psi(\alpha_1+1,\ \alpha_2,\ldots\alpha_{n-1},\ \alpha_n;\ \rho_1,\ \rho_2,\ldots\rho_n;\ x).$$

If then $R(s+1)$, $R(s+1,\ \alpha_n+1)$, denote respectively the remainders after $(s+1)$ terms of the right-hand member of equation (28) for the function involving $\alpha_1$, $\alpha_n$, and for that involving $\alpha_1$, $\alpha_n+1$, and if $R(s,\ \alpha_1+1)$ denote the remainder after $s$ terms for that involving $\alpha_1+1$, $\alpha_n$, we have

$$R(s+1,\ \alpha_n+1)=(\alpha_1-\alpha_n)\,R(s+1)-R(s,\ \alpha_1+1).$$

If then the theorem holds for $R(s,\ \alpha_n)$, a fortiori it holds for $R(s+1,\ \alpha_n)$, and it also holds for $R(s,\ \alpha_1+1)$, for $\sum\limits_{r=2}^{r=n}[\alpha_r-\rho_r]$ is not increased by increasing $\alpha_1$; we thus have

$$\text{mod. } R(s+1)<\text{mod.}\,\frac{\Pi(\alpha_1+s)\ldots\Pi(\alpha_1-\rho_n+1+s)}{\Pi(\alpha_1-\alpha_2+1+s)\ldots\Pi(\alpha_1-\alpha_n+1+s)}\,\frac{1}{\Pi(s+1)\,x^{\alpha_1+s+1}},$$

and mod. $R(s, \alpha_1+1) < (s+1)$ times the same, therefore provided $\alpha_1 - \alpha_n$ is positive we deduce that

$$\text{mod. } R(s+1, \alpha_n+1) < \text{mod. } \frac{\Pi(\alpha_1+s)\ldots\Pi(\alpha_1-\rho_n+1+s)}{\Pi(\alpha_1-\alpha_2+1+s)\ldots\Pi(\alpha_1-\alpha_{n-1}+1+s)\,\Pi(\alpha_1-\alpha_n+s)}\,\frac{1}{\Pi(s+1)\,x^{\alpha_1+s+1}},$$

which is the result stated; this reasoning holds for all values of $s$ including zero.

We will next show that if the theorem holds for the remainder after $s$ terms of the function involving $\alpha_1$, it also holds for the remainder after $(s+1)$ terms of the function involving $\alpha_1 - 1$, the other $\alpha$'s and $\rho$'s being unaltered.

If $\psi(\alpha_1)$ denote either the left-hand member of (28) or the sum of the terms at the beginning of the right up to and including the term involving a specified power of $x$, we have

$$x^{\alpha_1-2}\psi(\alpha_1) = \frac{d}{dx}\{x^{\alpha_1-1}\psi(\alpha_1-1)\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(55).$$

If then we denote the difference between the left-hand member of (28) and $s$ terms of the right-hand member by $R(s, \alpha_1, x)$, we have

$$x^{\alpha_1-2}R(s, \alpha_1, x) = \frac{d}{dx}\{x^{\alpha_1-1}R(s+1, \alpha_1-1, x)\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(56),$$

and therefore

$$\Big[x^{\alpha_1-1}R(s+1, \alpha_1-1, x)\Big]_x^\infty = \int_x^\infty x^{\alpha_1-2}R(s, \alpha_1, x)\,dx \quad\ldots\ldots\ldots\ldots\ldots\ldots(57).$$

For all values of $s$ including zero, $x^{\alpha_1-1}R(s+1, \alpha_1-1, x)$ vanishes when the modulus of $x$ is infinite, whatever be its argument, since equation (28) is true in the sense (B) and since the first term omitted when multiplied by $x^{\alpha_1-1}$ has for its index $\alpha_1+s$, the being negative of which is one of the conditions that the theorem should hold for $R(s, \alpha_1, x)$.

Accordingly

$$R(s+1, \alpha_1-1, x) = -x^{1-\alpha_1}\int_x^\infty x^{\alpha_1-2}R(s, \alpha_1, x)\,dx \quad\ldots\ldots\ldots\ldots\ldots\ldots(58).$$

By taking for the path of integration the production of the straight line joining the origin to the point $x$, it appears that either member of (58) would be increased in absolute value by replacing $R(s, \alpha_1, x)$ by the modulus of the first term omitted; but this change would replace the right-hand member by the modulus of the $(s+2)$th term in the series obtained from the right-hand member of (28) by diminishing $\alpha_1$ by unity; therefore $R(s+1, \alpha_1-1, x)$ is less in absolute value than the next term; this reasoning holds for all values of $s$ including zero.

We will now, having in fact proved the theorem for all values of $s$ including zero subject to the conditions that for all values of $r$ from 2 to $n$ inclusive $\alpha_1 - \rho_r + 1$ and $\rho_r - \alpha_r$ should be positive, proceed to examine what restrictions should be placed on $s$ if these conditions are violated.

We suppose that $\alpha_n$ is the greatest $\alpha$ and $\rho_n$ the greatest $\rho$.

Suppose first that $\alpha_1 - \rho_n + 1$ is negative lying between $-a_n$ and $-a_n + 1$, and that $\rho_n - \alpha_n$ is negative lying between $-b_n$ and $-b_n + 1$. Then $\alpha_1 - \alpha_n + 1$ must also be negative lying between $-c_n$ and $-c_n + 1$ where $c_n$ is either $a_n + b_n$ or $a_n + b_n - 1$. For the other values of $r$ from 2 to $n-1$ let $b_r$ denote $[\alpha_r - \rho_r]$, which for some values of $r$ may be zero.

The theorem then applies for all values of $s$ to the function involving

$$\alpha_1 + c_n,\ \alpha_2 - b_2,\ \dots\ \alpha_n - b_n,\ 1,\ \rho_1,\ \rho_2,\ \dots\ \rho_n,$$

since the necessary conditions are satisfied.

We may increase the value of $\alpha_n - b_n$ by unity $b_n$ times in succession, keeping all the other $\alpha$'s and $\rho$'s unaltered, provided at each such operation we increase the lowest value of $s$ for which the theorem holds by unity, the condition for the validity of the last such process being that $\alpha_1 + c_n - (\alpha_n - 1)$ is to be positive; thus when we attain the value $\alpha_n$, $s$ has to be raised from zero to $b_n$. Then for each other value of $r$ in turn we may increase in a similar manner the value of $\alpha_r - b_r$ by $b_r$, increasing the value of $s$ at the same time by $b_r$ also, the condition for the validity of this being that $\alpha_1 + c_n - (\alpha_r - 1)$ should be positive, which is true since $\alpha_n$ is the largest $\alpha$. Thus when we attain the values $\alpha_2, \alpha_3, \dots \alpha_n$, the lowest admissible value of $s$ is $\sum_{r=2}^{r=n} b_r$. Finally we diminish the value of $\alpha_1 + c_n$ by unity $c_n$ times in succession without altering the other $\alpha$'s or $\rho$'s, at the same time increasing the value of $s$ by $c_n$. Thus the lowest admissible value of $s$ is $[\alpha_n - \alpha_1 - 1] + \sum_2^n [\alpha_r - \rho_r]$, as the enunciation states.

Next suppose that $\alpha_1 - \rho_n + 1$ is positive but that $\rho_n - \alpha_n$ is negative; we have now two sub-cases according as $\alpha_1 - \alpha_n + 1$ is negative or positive.

Taking first the former: as before the theorem applies for all values of $s$ to the function involving

$$\alpha_1 + c_n,\ \alpha_2 - b_2,\ \dots\ \alpha_n - b_n,\ 1,\ \rho_1,\ \rho_2,\ \dots\ \rho_n,$$

and we proceed as before, with the result that when we attain the function involving

$$\alpha_1,\ \alpha_2,\ \dots\ \alpha_n,\ 1,\ \rho_1,\ \dots\ \rho_n,$$

the lowest admissible value of $s$ is $[\alpha_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

Taking the latter sub-case, the theorem now applies for all values of $s$ to the function involving

$$\alpha_1,\ \alpha_2 - b_2,\ \dots\ \alpha_n - b_n,\ 1,\ \rho_1,\ \dots\ \rho_n;$$

as before, for each value of $r$ in turn we may increase the value of $\alpha_r - b_r$ by $b_r$, increasing the value of $s$ by $b_r$ also, this being legitimate since $\alpha_1 - \alpha_r + 1$ is positive; thus when we attain the values $\alpha_1, \alpha_2, \ldots \alpha_n, 1, \rho_1, \ldots \rho_n$, the lowest admissible value of $s$ is $\sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

Next suppose that $\alpha_1 - \rho_n + 1$ is positive and $\rho_n - \alpha_n$ positive, but that for some values of $r$, $\rho_r - \alpha_r$ is negative. This is similar to the sub-case last considered and as in it, the lowest admissible value of $s$ is $\sum_{r=2}^{r=n} [\alpha_r - \rho_r]$, the term $[\alpha_n - \rho_n]$ being however zero.

Finally suppose that $\alpha_1 - \rho_n + 1$ is negative, and $\rho_n - \alpha_n$ positive. The theorem applies for all values of $s$ to the function involving

$$\alpha_1 + [\rho_n - \alpha_1 - 1],\ \alpha_2 - b_2, \ldots \alpha_n,\ 1,\ \rho_1, \ldots \rho_n.$$

As before, for each value of $r$ we may increase $\alpha_r - b_r$ by $b_r$, increasing $s$ by $b_r$ at the same time, this being legitimate since $\alpha_1 + [\rho_n - \alpha_1 - 1] - \alpha_r + 1$ is positive, $\rho_n$ being greater than $\alpha_n$ and a fortiori than $\alpha_r$. Finally we reduce the value of $\alpha_1 + [\rho_n - \alpha_1 - 1]$ to $\alpha_1$ and thus when we attain the values $\alpha_1, \alpha_2, \ldots \alpha_n, 1, \rho_1, \ldots \rho_n$ the lowest admissible value of $s$ is $[\rho_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$, the term $[\alpha_n - \rho_n]$ being zero.

These several limits are all included by the statement that $s$ is not to be less than the greater of the two integers, $[\rho_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$, $[\alpha_n - \alpha_1 - 1] + \sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

11. We may in fact obtain a limit to the error even when the real part of $x$ is negative. The reasoning of Art. 3 suffices to show that for the function $x^{-\alpha_1}\phi(\alpha_1, \alpha_1 - \rho_1 + 1;\ +1/x)$ if the argument of $x$ be $\pm\gamma$, $\gamma$ being $< \pi/2$, the modulus of the remainder is less than that of the next term divided by $(\sin(\theta+\gamma))^{[\alpha_1]}(\cos\theta)^{\alpha_1 - \rho_1 + s + 1}$, where $\theta$ and $\theta + \gamma$ are each less than $\pi/2$; and by changing in the integral (37) the point $h$ to the point at infinity on the production of the line joining the points 0, $x$, we see that the same statement holds for the function of the $(n+1)$th order if for all values of $r$ from 2 to $n$, $\alpha_1 - \rho_r + 1$ and $\rho_r - \alpha_r$ are positive. Also a reference to the method by which these restrictions are removed shows that in the most general case the index of $\sin(\theta+\gamma)$ may be replaced by the greater of the integers $[\alpha_1] + [\alpha_n - \alpha_1 - 1]$, $[\alpha_1] + [\rho_n - \alpha_1 - 1]$, while that of $\cos\theta$ is left unaltered in form, affected only by the increase in $s$, $s$ being the number of terms taken and subject to the same restrictions as before. We must bear in mind that every $\rho$ is greater than 1, $\rho_n$ the greatest $\rho$, $\alpha_n$ the greatest $\alpha$, and $\rho_1$ the $\rho$ omitted from the sum $\sum_{r=2}^{r=n} [\alpha_r - \rho_r]$.

We may investigate the numerical value in the case of the semiconvergent series for the Bessel functions. In this case we may write $\alpha_1 = \frac{1}{2} - n$, $\rho_1 = 1 - 2n$. Hence $[\alpha_1]$ is 1, $\alpha_1 - \rho_1 + s + 1$ is $\frac{1}{2} + n + s$. The divisor is thus $\sin(\theta+\gamma)(\cos\theta)^{\frac{1}{2}+n+s}$; to make this as large as possible, $\theta$ should be nearly zero unless $\gamma$ be very small, and we deduce that the error is less than the next term divided by $\sin\gamma$; if $\gamma$ be very small,

the greatest value is greater than when $\gamma$ is zero, in which case it is $(\frac{1}{2}+n+s)^{\frac{1}{2}(\frac{1}{2}+n+s)}$. $(\frac{3}{2}+n+s)^{-\frac{1}{2}(\frac{3}{2}+n+s)}$; this tends to equality with $(\frac{1}{2}+n+s)^{-\frac{1}{2}}e^{-\frac{1}{2}}$, even for moderate values of $s$, and the error is thus less than the next term multiplied by a number which is nearly $(\frac{1}{2}+n+s)^{\frac{1}{2}}e^{\frac{1}{2}}$. The multiplier thus obtained when $\gamma$ is zero is considerably larger than that given by Weber (*Math. Annal.* XXXVII.) for all arguments, which is about $s^{\frac{1}{2}}\pi^{-\frac{1}{2}}\cos n\pi$.

12. By reasoning similar to that by which it is shown that $\underset{\alpha=\infty}{\text{Lt.}}(1-x/\alpha)^{\alpha}=e^{-x}$ we may show that by writing $x=y/h$, $\alpha=-h$ and increasing $h$ indefinitely we can diminish the number of $\alpha$'s in equation (28) successively by unity; we thus obtain very general results. From the theorem that as $\tau$ increases indefinitely the ratio of $\Pi(\tau)$ to $e^{-\tau}\tau^{\tau}\sqrt{2\tau\pi}$ has unity for its limit it follows that if $\alpha$ and $\beta$ are positive quantities and $\tau$ be increased indefinitely $\dfrac{\Pi(\tau-\alpha)}{\Pi(\tau-\beta)}\tau^{\alpha-\beta}$ has unity for its limit, and making use of this result we see that the general theorem may be written in the form that if $m \not> n$

$$\frac{\Pi(\alpha_1-1)\,\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_2)\,\Pi(-\alpha_3)\ldots\Pi(-\alpha_m)}\cdot F(\alpha_1,\alpha_2,\ldots\alpha_m;\ \rho_1,\rho_2,\ldots\rho_n;\ (-)^{n-m}x)$$

$$+\frac{\Pi(\alpha_1-\rho_1)\,\Pi(\rho_1-2)\,\Pi(\rho_1-\rho_2-1)\ldots\Pi(\rho_1-\rho_n-1)}{\Pi(\rho_1-\alpha_2-1)\,\Pi(\rho_1-\alpha_3-1)\ldots\Pi(\rho_1-\alpha_m-1)}x^{1-\rho_1}F(\alpha_1-\rho_1+1,\alpha_2-\rho_1+1,\ldots\alpha_m-\rho_1+1;$$
$$2-\rho_1,\rho_2-\rho_1+1,\ldots\rho_n-\rho_1+1;\ (-)^{n-m}x),$$

$$+(n-1)\text{ terms analogous to the last}$$

$$=\frac{\Pi(\alpha_1-1)\,\Pi(\alpha_1-\rho_1)\ldots\Pi(\alpha_1-\rho_n)}{\Pi(\alpha_1-\alpha_2)\,\Pi(\alpha_1-\alpha_3)\ldots\Pi(\alpha_1-\alpha_m)}x^{-\alpha_1}F(\alpha_1,\alpha_1-\rho_1+1,\ldots\alpha_1-\rho_n+1;\ \alpha_1-\alpha_2+1,\ldots$$
$$\alpha_1-\alpha_m+1;\ -1/x)\ldots\ldots\ldots\ldots(59),$$

in the same senses as those in which equation (28) is true.

# VII. *A semi-inverse method of solution of the equations of elasticity, and its application to certain cases of aeolotropic ellipsoids and cylinders.* By C. CHREE, Sc.D., F.R.S.

[*Received* 28 Jan. 1899. *Read* 20 February.]

CONTENTS.


| Sec. | §§ | |
|---|---|---|
| I. | 1—9. | Preliminary, and general formulae. |
| II. | 10—15. | Sphere of material symmetrical round an axis. |
| III. | 16—19. | Flat ellipsoid. |
| IV. | 20—21. | Thin elliptic disk rotating about the perpendicular to its plane through the centre. |
| V. | 22—25. | Thin rotating circular disk of material symmetrical round the axis. |
| VI. | 26—28. | Elongated ellipsoid. |
| VII. | 29—31. | Long elliptic cylinder rotating about its long axis. |
| VIII. | 32—36. | Long rotating circular cylinder of material symmetrical round the axis. |


## SECTION I.

### PRELIMINARY, AND GENERAL FORMULAE.

§ 1. IN two previous papers, here termed (A)* and (B)† for brevity, I developed a new method of treating the elastic solid equations for isotropic material, which led to a complete solution of the problem presented by an ellipsoid of uniform density $\rho$,

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \quad \ldots\ldots (1),$$

acted on by bodily forces derivable from a potential

$$V = \tfrac{1}{2}(Px^2 + Qy^2 + Rz^2),$$

and by normal surface forces $Sx^2 + Ty^2 + Uz^2$.

Here $x$, $y$, $z$ are ordinary Cartesian coordinates; while $P$, $Q$, $R$, $S$, $T$, $U$ are constants, any of which may be zero.

* *Proc. Royal Soc.* Vol. LVIII. p. 89.

† *Quarterly Journal*, Vol. XXVII. p. 338, 1895.

The bodily forces are such as the gravitational forces arising from the self-attraction of the ellipsoid, the tide generating influence of a distant body, or the 'centrifugal forces' due to rotation about a principal axis of the ellipsoid.

The present paper deals with the extension of the method to aeolotropic bodies.

§ 2. The most general kind of elastic solid possesses on the usual theory 21 independent elastic constants. The consideration of such material is laborious owing to the length of the expressions. I have thus considered in detail no case of greater complexity than that presented by material symmetrical with respect to three planes of elastic symmetry, coincident with the three principal planes of the ellipsoid.

In such material there are three principal Young's moduli $E_1$, $E_2$, $E_3$, relating to tractions parallel to the axes of $x$, $y$ and $z$ respectively. There are six corresponding Poisson's ratios $\eta_{12}$, $\eta_{21}$, etc. Here the first suffix gives the direction of the traction, the second that of the corresponding contraction. For instance, $\eta_{12}$ is the ratio of the contraction parallel to the $y$-axis to the extension parallel to the $x$-axis, in the case of traction parallel to the latter axis.

Between the six Poisson's ratios there exist the three relations

$$\eta_{12}/E_1 = \eta_{21}/E_2;\quad \eta_{13}/E_1 = \eta_{31}/E_3;\quad \eta_{23}/E_2 = \eta_{32}/E_3 \quad\text{.....(2)};$$

so that only three are really independent. There being nine independent elastic constants for this type of material, we shall take for our remaining three the principal slide coefficients $n_1$, $n_2$, $n_3$.

In the notation of Todhunter and Pearson's *History of Elasticity*, the strains are given in terms of the stresses by the following relations

$$\left.\begin{array}{lll} s_x = (\widehat{xx} - \eta_{12}\widehat{yy} - \eta_{13}\widehat{zz})/E_1; & s_y = (\widehat{yy} - \eta_{21}\widehat{xx} - \eta_{23}\widehat{zz})/E_2; & s_z = (\widehat{zz} - \eta_{31}\widehat{xx} - \eta_{32}\widehat{yy})/E_3; \\ \sigma_{yz} = \widehat{yz}/n_1 & ;\ \sigma_{zx} = \widehat{zx}/n_2 & ;\ \sigma_{xy} = \widehat{xy}/n_3 \end{array}\right\} \text{.....(3)}.$$

A second and simpler type of material dealt with here is that which in addition to symmetry with respect to the planes of $zx$, $xy$ and $yz$ is completely symmetrical round the axis of $z$.

For such material we have

$$\left.\begin{array}{r} E_2 = E_1 = E', \\ E_3 = E, \\ \eta_{31} = \eta_{32} = \eta, \\ \eta_{12} = \eta_{21} = \eta', \\ \eta_{13} = \eta_{23} = \eta'', \\ \eta''/E' = \eta/E, \\ n_2 = n_1 = n, \\ n_3 = E' \div \{2\,(1+\eta')\} \end{array}\right\} \text{.....(4)}.$$

There are in this case only five independent elastic constants.

§ 3. For any kind of elastic material the stresses must satisfy the three body-stress equations

$$\left.\begin{aligned}\frac{d\widehat{xx}}{dx}+\frac{d\widehat{xy}}{dy}+\frac{d\widehat{xz}}{dz}+\rho Px&=0,\\ \frac{d\widehat{xy}}{dx}+\frac{d\widehat{yy}}{dy}+\frac{d\widehat{yz}}{dz}+\rho Qy&=0,\\ \frac{d\widehat{xz}}{dx}+\frac{d\widehat{yz}}{dy}+\frac{d\widehat{zz}}{dz}+\rho Rz&=0\end{aligned}\right\}\quad\text{.....(5)},$$

and the three surface equations

$$\left.\begin{aligned}(x/a^2)\,\widehat{xx}+(y/b^2)\,\widehat{xy}+(z/c^2)\,\widehat{xz}&=(x/a^2)\,(Sx^2+Ty^2+Uz^2),\\ (x/a^2)\,\widehat{xy}+(y/b^2)\,\widehat{yy}+(z/c^2)\,\widehat{yz}&=(y/b^2)\,(Sx^2+Ty^2+Uz^2),\\ (x/a^2)\,\widehat{xz}+(y/b^2)\,\widehat{yz}+(z/c^2)\,\widehat{zz}&=(z/c^2)\,(Sx^2+Ty^2+Uz^2)\end{aligned}\right\}\quad\text{.....(6)}.$$

From the six equations of compatibility* between the strains, viz. three of the type

$$\frac{d^2s_x}{dy^2}+\frac{d^2s_y}{dx^2}-\frac{d^2\sigma_{xy}}{dx\,dy}=0\quad\text{.....(7)},$$

and three of the type

$$\frac{2d^2s_x}{dy\,dz}+\frac{d^2\sigma_{yz}}{dx^2}-\frac{d^2\sigma_{zx}}{dx\,dy}-\frac{d^2\sigma_{xy}}{dx\,dz}=0\quad\text{..... (8)},$$

we get six corresponding equations between the stresses.

These equations necessarily vary with the nature of the material. Thus for material symmetrical about the coordinate planes we have three equations of the type

$$\frac{1}{E_1}\frac{d^2}{dy^2}(\widehat{xx}-\eta_{12}\widehat{yy}-\eta_{13}\widehat{zz})+\frac{1}{E_2}\frac{d^2}{dx^2}(\widehat{yy}-\eta_{21}\widehat{xx}-\eta_{23}\widehat{zz})-\frac{1}{n_3}\frac{d^2}{dx\,dy}\widehat{xy}=0\quad\text{..... (9)},$$

and three of the type

$$\frac{2}{E_1}\frac{d^2}{dy\,dz}(\widehat{xx}-\eta_{12}\widehat{yy}-\eta_{13}\widehat{zz})+\frac{1}{n_1}\frac{d^2}{dx^2}\widehat{yz}-\frac{1}{n_2}\frac{d^2}{dx\,dy}\widehat{xz}-\frac{1}{n_3}\frac{d^2}{dx\,dz}\widehat{xy}=0\quad\text{..... (10)}.$$

In the present instance the equations of type (10) are identically satisfied and need not concern us further.

When the material is symmetrical round the $z$-axis we find in place of (9)

$$\left.\begin{aligned}\frac{d^2}{dy^2}\left(\frac{\widehat{xx}-\eta'\widehat{yy}}{E'}-\frac{\eta}{E}\widehat{zz}\right)+\frac{d^2}{dx^2}\left(\frac{\widehat{yy}-\eta'\widehat{xx}}{E'}-\frac{\eta}{E}\widehat{zz}\right)-\frac{2\,(1+\eta')}{E'}\frac{d^2}{dx\,dy}\widehat{xy}&=0,\\ \frac{d^2}{dz^2}\left(\frac{\widehat{yy}-\eta'\widehat{xx}}{E'}-\frac{\eta}{E}\widehat{zz}\right)+\frac{1}{E}\frac{d^2}{dy^2}(\widehat{zz}-\eta\widehat{xx}-\eta\widehat{yy})-\frac{1}{n}\frac{d^2}{dy\,dz}\widehat{yz}&=0,\\ \frac{d^2}{dz^2}\left(\frac{\widehat{xx}-\eta'\widehat{yy}}{E'}-\frac{\eta}{E}\widehat{zz}\right)+\frac{1}{E}\frac{d^2}{dx^2}(\widehat{zz}-\eta\widehat{xx}-\eta\widehat{yy})-\frac{1}{n}\frac{d^2}{dx\,dz}\widehat{xz}&=0\end{aligned}\right\}\quad\text{..... (11)}.$$

* See Todhunter and Pearson's *History of Elasticity*, Vol. II. Part i. p. 74; or Love's *Treatise on Elasticity*, Vol. I. p. 122.

§ 4. The greater complexity of (9) or (11), as compared to the corresponding equations for isotropy, does not affect the *type* of solution; and, as in the papers (A) and (B), we may assume

$$\left.\begin{aligned}
\widehat{xx} &= A_0 + A_2x^2 + A_2'y^2 + A_2''z^2,\\
\widehat{yy} &= B_0 + B_2x^2 + B_2'y^2 + B_2''z^2,\\
\widehat{zz} &= C_0 + C_2x^2 + C_2'y^2 + C_2''z^2,\\
\widehat{yz} &= 2Lyz, \quad \widehat{xz} = 2Mxz, \quad \widehat{xy} = 2Nxy
\end{aligned}\right\} \quad \text{.....(12).}$$

Here $A_0, A_2 \ldots N$ are constants to be determined from the body-stress equations (5), the surface equations (6), and the equations of compatibility, the latter of which alone vary with the type of material.

Fortunately there is an immense economy of labour owing to my having in papers (A) and (B) expressed all the $A$, $B$, $C$ constants in (12) in terms of the three $L$, $M$, $N$. In effecting this simplification I employed only (5) and (6), equations which, as pointed out above, apply to all kinds of elastic material. We are thus enabled at once to replace (12) by the following equations established in the two earlier papers:—

$$\left.\begin{aligned}
\widehat{xx} &= Sx^2 + Ty^2 + Uz^2 + a^2\left[(\tfrac{1}{2}P\rho + S)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right.\\
&\qquad \left. + M\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right) + N\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right)\right],\\
\widehat{yy} &= Sx^2 + Ty^2 + Uz^2 + b^2\left[(\tfrac{1}{2}Q\rho + T)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right.\\
&\qquad \left. + N\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) + L\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right)\right],\\
\widehat{zz} &= Sx^2 + Ty^2 + Uz^2 + c^2\left[(\tfrac{1}{2}R\rho + U)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right.\\
&\qquad \left. + L\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right) + M\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\right],\\
\widehat{yz} &= 2Lyz, \quad \widehat{xz} = 2Mxz, \quad \widehat{xy} = 2Nxy
\end{aligned}\right\} \quad \text{.....(13).}$$

§ 5. The results (13) apply to all kinds of elastic material, whether possessed of 2 or of 21 independent elastic constants; but the values of $L$, $M$, $N$ vary of course with the material. The expressions for the strains corresponding to (13) vary. Thus, for material symmetrical with respect to the three coordinate planes, we find from (3)

$$\left.\begin{aligned}
s_xE_1 &= (1 - \eta_{12} - \eta_{13})(Sx^2 + Ty^2 + Uz^2) + \{a^2(\tfrac{1}{2}P\rho + S) - \eta_{12}b^2(\tfrac{1}{2}Q\rho + T)\\
&\qquad - \eta_{13}c^2(\tfrac{1}{2}R\rho + U)\}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)\\
&\quad + (a^2M - \eta_{12}b^2L)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right) + (a^2N - \eta_{13}c^2L)\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right)\\
&\quad - (\eta_{12}b^2N + \eta_{13}c^2M)\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right),\\
\sigma_{yz} &= 2Lyz/n_1
\end{aligned}\right\} \quad \text{.....(14).}$$

The expressions for the other four strains may be written down from symmetry.

If the material be symmetrical round the $z$-axis we have

$$\left.\begin{aligned}
s_x &= \left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)(Sx^2 + Ty^2 + Uz^2) + \left\{\frac{1}{E'}\left(a^2(\tfrac{1}{2}P\rho + S) - \eta' b^2(\tfrac{1}{2}Q\rho + T)\right)\right. \\
&\qquad\qquad \left. - \frac{\eta}{E}c^2(\tfrac{1}{2}R\rho + U)\right\}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \\
&\quad + \left(\frac{a^2M - \eta' b^2 L}{E'}\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right) + \left(\frac{a^2N}{E'} - \frac{\eta c^2 L}{E}\right)\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right) \\
&\quad - \left(\frac{\eta'}{E'}b^2N + \frac{\eta}{E}c^2M\right)\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right), \\
s_y &= \text{expression obtained from that for } s_x \text{ by interchanging } a \text{ with } b,\ x \text{ with} \\
&\qquad y,\ L \text{ with } M,\ P \text{ with } Q, \text{ and } S \text{ with } T, \\
Es_z &= (1-2\eta)(Sx^2 + Ty^2 + Uz^2) + \{c^2(\tfrac{1}{2}R\rho + U) - \eta a^2(\tfrac{1}{2}P\rho + S) - \eta b^2(\tfrac{1}{2}Q\rho + T)\} \\
&\qquad\qquad \times\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \\
&\quad - \eta(a^2M + b^2L)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{3z^2}{c^2}\right) + (c^2L - \eta a^2N)\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2} - \frac{z^2}{c^2}\right) \\
&\quad + (c^2M - \eta b^2N)\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right), \\
\sigma_{yz} &= 2Lyz/n, \quad \sigma_{zx} = 2Mxz/n, \quad \sigma_{xy} = \tfrac{4}{3}(1+\eta')Nxy/E'
\end{aligned}\right\} \quad \ldots\ldots(15).$$

§ 6. Results which depend only on the form of equations (13) are true irrespective of the nature of the elastic material. For instance* if $S$, $T$ and $U$ vanish, or there be no surface forces, the resultant stresses across parallel tangent planes at their points of contact with the system of confocals

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = \lambda$$

are all parallel, and their intensity varies as $1-\lambda$.

§ 7. We have now to consider how $L$, $M$ and $N$ are to be determined. Substituting from (13) in the equations of compatibility, whether (9) or (11) as the case may be, we obtain three simple equations of the form

$$\left.\begin{aligned}
a_{11}L + a_{12}M + a_{13}N &= \varpi_1, \\
a_{12}L + a_{22}M + a_{23}N &= \varpi_2, \\
a_{13}L + a_{23}M + a_{33}N &= \varpi_3
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

where $a_{11}, \ldots \varpi_1 \ldots$ are known functions of the elastic constants, the bodily and surface forces, and the semi-axes of the ellipsoid. Representing by $\Pi_{11}$, $\Pi_{12}$, &c. the minors of the determinant

$$\Pi \equiv \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{12}, & a_{22}, & a_{23} \\ a_{13}, & a_{23}, & a_{33} \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

* Cf. (A) § 2.

we have from (16)

$$\left.\begin{aligned} L &= (\varpi_1\Pi_{11} + \varpi_2\Pi_{12} + \varpi_3\Pi_{13}) \div \Pi, \\ M &= (\varpi_1\Pi_{12} + \varpi_2\Pi_{22} + \varpi_3\Pi_{23}) \div \Pi, \\ N &= (\varpi_1\Pi_{13} + \varpi_2\Pi_{23} + \varpi_3\Pi_{33}) \div \Pi \end{aligned}\right\} \quad \text{.....(18).}$$

§ 8. When the material is symmetrical with respect to the three principal planes of the ellipsoid

$$\left.\begin{aligned} a_{11} &= \frac{3b^4}{E_2} + \frac{3c^4}{E_3} + b^2c^2\left(\frac{1}{n_1} - \frac{2\eta_{23}}{E_2}\right), \\ a_{23} &= \frac{a^4}{E_1} - a^2b^2\frac{\eta_{12}}{E_1} - 3b^2c^2\frac{\eta_{23}}{E_2} - c^2a^2\frac{\eta_{31}}{E_3} \end{aligned}\right\} \quad \text{.....(19),}$$

and the other $a$'s can be written down from symmetry, the relations (4) being borne in mind.

For the same kind of material

$$\left.\begin{aligned} \varpi_1 &= (\tfrac{1}{2}P\rho + S)\frac{a^2}{E_1}(\eta_{12}b^2 + \eta_{13}c^2) + \frac{b^2}{E_2}\left[\tfrac{1}{2}Q\rho(\eta_{23}c^2 - b^2) + T\left\{c^2(1-\eta_{31})\frac{E_2}{E_3} - b^2\right\}\right] \\ &\quad + \frac{c^2}{E_3}\left[\tfrac{1}{2}R\rho(\eta_{32}b^2 - c^2) + U\left\{b^2(1-\eta_{21})\frac{E_3}{E_2} - c^2\right\}\right], \\ \varpi_2 &= \frac{a^2}{E_1}\left[\tfrac{1}{2}P\rho(\eta_{13}c^2 - a^2) + S\left\{c^2(1-\eta_{32})\frac{E_1}{E_3} - a^2\right\}\right] + (\tfrac{1}{2}Q\rho + T)\frac{b^2}{E_2}(\eta_{23}c^2 + \eta_{21}a^2) \\ &\quad + \frac{c^2}{E_3}\left[\tfrac{1}{2}R\rho(\eta_{31}a^2 - c^2) + U\left\{a^2(1-\eta_{12})\frac{E_3}{E_1} - c^2\right\}\right], \\ \varpi_3 &= \frac{a^2}{E_1}\left[\tfrac{1}{2}P\rho(\eta_{12}b^2 - a^2) + S\left\{b^2(1-\eta_{23})\frac{E_1}{E_2} - a^2\right\}\right] \\ &\quad + \frac{b^2}{E_2}\left[\tfrac{1}{2}Q\rho(\eta_{21}a^2 - b^2) + T\left\{a^2(1-\eta_{13})\frac{E_2}{E_1} - b^2\right\}\right] + (\tfrac{1}{2}R\rho + U)\frac{c^2}{E_3}(\eta_{31}a^2 + \eta_{32}b^2) \end{aligned}\right\} \quad \text{.....(20).}$$

Under like conditions

$$\left.\begin{aligned} \Pi_{11} &= 8\frac{a^8}{E_1^2} + \frac{a^4b^4}{E_1E_2}(9 - \eta_{12}\eta_{21}) + \frac{a^4c^4}{E_1E_3}(9 - \eta_{13}\eta_{31}) + 9\frac{b^4c^4}{E_2E_3}(1 - \eta_{23}\eta_{32}) \\ &\quad + \frac{a^6b^2}{E_1}\left(\frac{3}{n_2} - \frac{4\eta_{12}}{E_1}\right) + \frac{a^6c^2}{E_1}\left(\frac{3}{n_3} - \frac{4\eta_{13}}{E_1}\right) + a^4b^2c^2\left\{\left(\frac{1}{n_2} - \frac{2\eta_{12}}{E_1}\right)\left(\frac{1}{n_3} - \frac{2\eta_{13}}{E_1}\right) + \frac{6\eta_{23}}{E_1E_2} - \frac{2\eta_{12}\eta_{13}}{E_1^2}\right\} \\ &\quad + 3\frac{a^2b^4c^2}{E_2}\left(\frac{1}{n_2} - \frac{2\eta_{13}}{E_1} - \frac{2\eta_{12}\eta_{23}}{E_1}\right) + 3\frac{a^2b^2c^4}{E_3}\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1} - \frac{2\eta_{13}\eta_{32}}{E_1}\right), \\ \Pi_{23} &= \frac{b^4c^4}{E_2}\left\{\frac{1}{E_3}(1 - 5\eta_{23}\eta_{32}) + \frac{3\eta_{23}}{n_1}\right\} - 3\frac{a^4b^4}{E_1E_2}(1 - \eta_{12}\eta_{21}) - 3\frac{a^4c^4}{E_1E_3}(1 - \eta_{13}\eta_{31}) \\ &\quad + 8\frac{b^2c^2}{E_2E_3}(\eta_{32}b^4 + \eta_{23}c^4) + \frac{a^2b^4c^2}{E_1}\left(\frac{2\eta_{13}}{E_2} + \frac{2\eta_{12}\eta_{23}}{E_2} + \frac{\eta_{12}}{n_1}\right) \\ &\quad + \frac{a^2b^2c^4}{E_1}\left(\frac{2\eta_{12}}{E_3} + \frac{2\eta_{13}\eta_{32}}{E_3} + \frac{\eta_{13}}{n_1}\right) + \frac{a^4b^2c^2}{E_1}\left(\frac{2\eta_{23}}{E_2} + \frac{10\eta_{12}\eta_{13}}{E_1} - \frac{1}{n_1}\right) \end{aligned}\right\} \quad \text{...(21).}$$

The other four minors of the determinant (17) may be written down from symmetry.

In obtaining (20) and (21) free use has been made of the relations (2), by means of which various alternative forms can be obtained.

The full expression for $\Pi$ is too long to write down. In practice, after determining the minors as above, one would determine $\Pi$ from such an equation as

$$\Pi = a_{11}\Pi_{11} + a_{12}\Pi_{12} + a_{13}\Pi_{13} \quad \text{.....(22).}$$

§ 9. When the material is symmetrical round the $z$-axis

$$\left.\begin{aligned}
a_{11} &= 3\left(\frac{b^4}{E'} + \frac{c^4}{E}\right) + b^2c^2\left(\frac{1}{n} - \frac{2\eta}{E}\right),\\
a_{22} &= 3\left(\frac{a^4}{E'} + \frac{c^4}{E}\right) + c^2a^2\left(\frac{1}{n} - \frac{2\eta}{E}\right),\\
a_{33} &= (3a^4 + 2a^2b^2 + 3b^4)/E',\\
a_{12} &= \frac{c^2}{E}\{c^2 - \eta(a^2 + b^2)\} - 3a^2b^2\frac{\eta'}{E'},\\
a_{13} &= \frac{b^2}{E'}(b^2 - \eta' a^2) - c^2\frac{\eta}{E}(3a^2 + b^2),\\
a_{23} &= \frac{a^2}{E'}(a^2 - \eta' b^2) - c^2\frac{\eta}{E}(a^2 + 3b^2)
\end{aligned}\right\} \quad \text{.....(23).}$$

The corresponding $\varpi$'s and $\Pi$'s may be obtained from (20) and (21) by making use of (4); it would occupy space unduly to record them in full. In particular cases it is frequently simpler to employ the primitive equations (16) than to substitute their values for the $\varpi$'s and $\Pi$'s in (18).

My present object is rather to exemplify the utility of the method than to accumulate lengthy expressions, complete from a mathematical standpoint but indigestible by the ordinary physicist; I thus proceed to the consideration of some special cases.

## SECTION II.

### Sphere of Material symmetrical round an Axis.

§ 10. Let us first consider the effect of mutual gravitation of the material. We have

$$S = T = U = 0,$$
$$P = Q = R = -g/a,$$

where $a$ is the radius of the sphere, $g$ 'gravity' at its surface.

The equations (16) take the form

$$\left.\begin{aligned}
a_{11}L + a_{12}M + a_{13}N = a_{12}L + a_{22}M + a_{23}N &= \tfrac{1}{2}g\rho a^2\left(\frac{1-3\eta}{E} + \frac{1-\eta'}{E'}\right),\\
a_{13}L + a_{23}M + a_{33}N &= g\rho a^2\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)
\end{aligned}\right\} \quad \text{..... (24);}$$

where $a_{11}$, &c. are obtained by putting $b = c = a$ in (23).

The formal solutions of (24) may conveniently be written

$$\left.\begin{aligned} L\Pi &= \tfrac{1}{2}g\rho a^3\left[\left(\frac{1-3\eta}{E}+\frac{1-\eta'}{E'}\right)(\Pi_{11}+\Pi_{12})+2\left(\frac{1-\eta'}{E'}-\frac{\eta}{E}\right)\Pi_{13}\right],\\ M\Pi &= \tfrac{1}{2}g\rho a^3\left[\left(\frac{1-3\eta}{E}+\frac{1-\eta'}{E'}\right)(\Pi_{12}+\Pi_{22})+2\left(\frac{1-\eta'}{E'}-\frac{\eta}{E}\right)\Pi_{23}\right],\\ N\Pi &= \tfrac{1}{2}g\rho a^3\left[\left(\frac{1-3\eta}{E}+\frac{1-\eta'}{E'}\right)(\Pi_{13}+\Pi_{23})+2\left(\frac{1-\eta'}{E'}-\frac{\eta}{E}\right)\Pi_{33}\right]\end{aligned}\right\}\text{.....}\ (25).$$

Here it is easily seen that

$$a_{22}=a_{11},\quad a_{23}=a_{13},$$
$$\Pi_{22}=\Pi_{11},\ \Pi_{23}=\Pi_{13},$$

and so

$$M=L.$$

It is also easily verified that

$$\left.\begin{aligned}(\Pi_{11}+\Pi_{12})\div(8/E') &= \Pi_{13}\div\left(\frac{4\eta}{E}-\frac{1-\eta'}{E'}\right),\\ &= \Pi\div\left[2a^4\left\{-\frac{16\eta^2}{E^2}+\frac{8(2-\eta-\eta\eta')}{EE'}+\frac{(1-\eta')(11+\eta')}{E'^2}+\frac{4}{nE'}\right\}\right],\\ &= a^6\left\{\frac{2}{E}+\frac{3(1+\eta')}{E'}+\frac{1}{n}\right\}\end{aligned}\right\}\text{...}\ (26).$$

Thus we can get rid of the common factor $a^6\left\{\frac{2}{E}+\frac{3(1+\eta')}{E'}+\frac{1}{n}\right\}$, which is an immense simplification.

In passing, the following simple way of obtaining the value of $\Pi$ may be noted. We have

$$\begin{aligned}\Pi &= a_{11}\Pi_{11}+a_{12}\Pi_{12}+a_{13}\Pi_{13}\\ &= (a_{11}-a_{12})(\Pi_{11}-\Pi_{12})+(a_{11}\Pi_{12}+a_{12}\Pi_{11}+a_{13}\Pi_{13}).\end{aligned}$$

But

$$a_{11}=a_{22},\ \text{and}\ a_{13}=a_{23},$$

so

$$a_{11}\Pi_{12}+a_{12}\Pi_{11}+a_{13}\Pi_{13}=a_{12}\Pi_{11}+a_{22}\Pi_{12}+a_{23}\Pi_{13}=0,$$

as being the value of a determinant of which two columns are identical.

Hence

$$\Pi=(a_{11}-a_{12})(\Pi_{11}-\Pi_{12})\text{.....}\ (27).$$

Returning to (25) we find

$$\left.\begin{aligned}L=M &= \frac{1}{2}\frac{g\rho}{a\Pi'}\left\{\frac{4(1-2\eta)}{EE'}+\left(\frac{1-\eta'}{E'}-\frac{\eta}{E}\right)\left(\frac{3+\eta'}{E'}+\frac{4\eta}{E}\right)\right\},\\ N &= \frac{1}{2}\frac{g\rho}{a\Pi'}\left[\frac{3(1-2\eta)(1-\eta')}{EE'}+2\left(\frac{1-\eta'}{E'}-\frac{\eta}{E}\right)\left(\frac{1-\eta'}{E'}+\frac{4\eta}{E}+\frac{1}{2n}\right)\right]\end{aligned}\right\}\text{.....}(28),$$

where

$$\Pi'\equiv\frac{-16\eta^2}{E^2}+\frac{8}{EE'}(2-\eta-\eta\eta')+\frac{(1-\eta')(11+\eta')}{E'^2}+\frac{4}{nE'}\text{.....}\ (29).$$

Employing these values of $L$, $M$ and $N$, with

$$S = T = U = 0, \text{ and } P = Q = R = -g/a,$$

we have the stresses given by (13) and the strains by (15).

§ 11. It is often best to retain the general formulae (13) and (15) as long as possible, and only substitute for $L$, $M$ and $N$ in the final result. Suppose, for instance, we wish to find the change of length in radii along and perpendicular to the axis of material symmetry, due to the gravitational forces. Let $\delta a_1$ represent the change of radius along $ox$ (and so perpendicular to the axis of symmetry), $\delta a_3$ the change along $oz$. Then, remembering that

$$M = L,$$

we have from (15)

$$\delta a_1 = \int_0^a s_x dx = -\frac{a}{3}\left\{\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)(g\rho a - 2La^2) - \frac{2N}{E'}a^2\right\},$$

$$\delta a_3 = \int_0^a s_z dz = -\frac{a}{3E}\{(1-2\eta)\,g\rho a - 4(L-\eta N)\,a^2\}.$$

After reduction I find

$$\delta a_1/a = -\frac{g\rho a}{\Pi'}\left[\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)\left\{-\frac{4\eta^2}{E^2} + \frac{4-3\eta-\eta\eta'}{EE'} + \frac{2(1-\eta')}{E'^2} + \frac{1}{nE'}\right\} - (1-2\eta)\frac{1-\eta'}{EE'^2}\right] \ldots (30),$$

$$\delta a_3/a = -\frac{1}{3}\frac{g\rho a}{E\Pi'}\left[(1-2\eta)\left\{-\frac{8\eta^2}{E^2} + \frac{2(4-5\eta-3\eta\eta')}{EE'} + \frac{(1-\eta')(11+\eta')}{E'^2} + \frac{4}{nE'}\right\}\right.$$
$$\left. - 2\left(\frac{1-\eta'}{E'} - \frac{\eta}{E}\right)\left(\frac{3-2\eta+\eta'+2\eta\eta'}{E'} - \frac{\eta}{n}\right)\right] \ldots\ldots\ldots\ldots (31);$$

where $\Pi'$ is given by (29).

There is, as will presently appear, a reason for not multiplying up by the factors $1-2\eta$ and $\frac{1-\eta'}{E'} - \frac{\eta}{E}$.

§ 12. If we suppose $E'/E$ very small, or that the stretching of a bar under given longitudinal traction is very much less when its axis is along than when it is perpendicular to the axis of material symmetry, we have as first approximations

$$\left.\begin{aligned} \delta a_1/a &= -g\rho a(1-\eta')\left\{2\frac{1-\eta'}{E'} + \frac{1}{n}\right\} \div (E'^2\Pi'), \\ \delta a_3/\delta a_1 &= \text{expression of order } E'/E \end{aligned}\right\} \ldots\ldots\ldots\ldots (32).$$

The change of length of the diameter along the axis of symmetry, which is here the direction of high resistance to extension, is thus relatively negligible.

If on the other hand $E/E'$ be very small, we find as first approximations

$$\Pi' = -16\eta^2/E^2 \ldots\ldots\ldots\ldots (33),$$

$$\left.\begin{aligned} \delta a_1/a &= g\rho a\eta/(4E), \\ \delta a_3/a &= -\tfrac{1}{24}g\rho a\left\{\frac{4(1-2\eta)}{E} + \frac{1}{n}\right\} \end{aligned}\right\} \ldots\ldots\ldots\ldots (34).$$

In this case $\delta a_1$ and $\delta a_3$ are in general of the same order of magnitude, only $\delta a_1$ is absolutely positive.

If while $E$ and $E'$ are of the same order of magnitude $\eta$ vanishes, or traction parallel to the axis of material symmetry causes no lateral contraction in a bar, we have

$$\delta a_1/a = -g\rho a(1-\eta')\left\{\frac{3}{E}+\frac{2(1-\eta')}{E'}+\frac{1}{n}\right\} \div (\Pi' E'^2) \quad \text{.....(35)},$$

$$\delta a_3/a = -g\rho a\left\{\frac{8}{E}+\frac{(1-\eta')(5-\eta')}{E'}+\frac{4}{n}\right\} \div (3\Pi' E E') \quad \text{.....(36)},$$

with

$$\Pi' = \frac{1}{E'}\left\{\frac{16}{E}+\frac{(1-\eta')(11+\eta')}{E'}+\frac{4}{n}\right\} \quad \text{.....(37)}.$$

Both (35) and (36) show contraction of diameters.

§ 13. If we apply our formulae to a sphere of the earth's mass the numerical results, when we attribute any ordinary values to the elastic constants, are inconsistent with the fundamental assumption of the ordinary mathematical theory of elasticity, according to which strains are small quantities whose squares are negligible.

In an isotropic "earth" consistency is attained only by supposing the material to be nearly incompressible, or $(1-2\eta)/E$ very nearly zero. This happens only if $\eta$ be nearly ·5, unless $E$ be enormously greater than for any known material. The former alternative is much the less improbable, because it implies that whilst resistance to change of volume is enormous—as it may well be under enormous pressure—resistance to change of shape need not be excessive.

When the material is not isotropic, but of the 5 elastic constant type symmetrical round an axis, the ordinary criterion for incompressibility is

$$2\left(\frac{1-\eta'}{E'}-\frac{\eta}{E}\right)+\frac{1-2\eta}{E}=0 \quad \text{.....(38)}.$$

This ensures that no change of volume will follow the application of *uniform* pressure however large; but it does not prevent change of volume under pressure which is not uniform. To provide against any change of volume two independent conditions must be satisfied, viz.

$$1-2\eta=0 \quad \text{.....(39)},$$

$$\frac{1-\eta'}{E'}-\frac{\eta}{E}=0 \quad \text{.....(40)}.$$

If these hold simultaneously, of course (38) holds likewise. If (39) and (40) both hold, then (30) and (31) show at once that $\delta a_1$ and $\delta a_3$ absolutely vanish. If however (38) holds alone, or if one only of the two conditions (39) and (40) is satisfied, then neither $\delta a_1$ nor $\delta a_3$ vanishes.

§ 14. As the case when the material is only nearly incompressible is of special interest, it merits our attention.

Let us suppose then

$$\left.\begin{aligned} 1-2\eta &= p, \\ (1-\eta')/E' - \eta/E &= q/E \end{aligned}\right\} \quad \text{.....(41)},$$

where $p$ and $q$ are very small.

We easily reduce (30) and (31) to

$$\left.\begin{aligned} \delta a_1/a &= -\frac{g\rho a}{E\Pi'}\left\{q\left(\frac{3}{EE'}-\frac{3}{4E^2}+\frac{1}{nE'}\right)-\frac{1}{2}\frac{p}{EE'}\right\}, \\ \delta a_2/a &= -\frac{1}{3}\frac{g\rho a}{E\Pi'}\left\{p\left(\frac{6}{EE'}-\frac{3}{4E^2}+\frac{4}{nE'}\right)-2\frac{q}{E}\left(\frac{4}{E'}-\frac{1}{E}-\frac{1}{2n}\right)\right\} \end{aligned}\right\} \text{.....(42)};$$

where $p^2$, $pq$ and $q^2$ are neglected, and

$$\Pi' = \frac{14}{EE'}-\frac{9}{4E^2}+\frac{4}{nE'} \quad \text{.....(43)}.$$

If, in addition, the material be absolutely incompressible under uniform pressure,

$$p+2q=0,$$

and so

$$\left.\begin{aligned} \delta a_1/a &= \frac{g\rho a p}{2E\Pi'}\left(\frac{4}{EE'}-\frac{3}{4E^2}+\frac{1}{nE'}\right), \\ \delta a_2/a &= -\frac{g\rho a p}{3E\Pi'}\left(\frac{10}{EE'}-\frac{7}{4E^2}+\frac{4}{nE'}-\frac{1}{2nE}\right) \end{aligned}\right\} \text{.....(44)}.$$

Unless $E$ and $E'$ are widely different in magnitude, $\delta a_1$ has here the same sign as $p$, while $\delta a_2$ has the opposite sign.

These illustrations will, I hope, suffice to show how very varied are the possibilities in a gravitating elastic solid "earth."

## ROTATING SPHERE OF MATERIAL SYMMETRICAL ROUND AN AXIS.

§ 15. The discussion of the influence of gravitation on an elastic solid earth naturally suggests that of rotation.

I have already* considered the influence of rotation on a spheroid of material symmetrical round the axis of rotation; and shall thus merely write down, for comparison with (30) and (31), the expressions for the changes in the equatorial and polar semi-axes of a sphere. These expressions may be obtained from the formulae (96) and (97) of the paper just quoted.

* *Camb. Phil. Trans.* Vol. xv. pp. 1—36.

Employing $\Pi'$ as in (29), and denoting the angular velocity by $\omega$, I find

$$\delta a_1/a = \frac{\omega^2\rho a^2(1-\eta')}{E'\Pi'}\left[\left\{-\frac{4\eta^2}{E^2}+\frac{4-3\eta-\eta\eta'}{EE'}+\frac{2(1-\eta')}{E'^2}+\frac{1}{nE'}\right\}+\frac{2\eta}{EE'}\right] \quad \text{.....(45)},$$

$$\delta a_3/a = -\frac{2\omega^2\rho a^2}{3E\Pi'}\left[\eta\left\{-\frac{8\eta^2}{E^2}+\frac{2(4-5\eta-3\eta\eta')}{EE'}+\frac{(1-\eta')(11+\eta')}{E'^2}+\frac{4}{nE'}\right\}\right.$$
$$\left.+\frac{1-\eta'}{E'}\left(\frac{3-2\eta+\eta'+2\eta\eta'}{E'}-\frac{\eta}{n}\right)\right] \quad \text{.....(46)}.$$

I have manipulated these expressions so as to facilitate comparison with the corresponding results (30) and (31) for the influence of gravitation.

If in (45) and (46) we suppose $E'/E$ very small, we have

$$\left.\begin{aligned}\delta a_1/a &= \frac{\omega^2\rho a^2(1-\eta')}{E'^2\Pi'}\left\{\frac{2(1-\eta')}{E'}+\frac{1}{n}\right\},\\ \delta a_3/\delta a_1 &= \text{expression of order } E'/E \text{ and so negligible}\end{aligned}\right\} \quad \text{.....(47)}.$$

The similarity with the corresponding results (32) for gravitation is noteworthy.

If on the other hand $E/E'$ be very small, we find, remembering that $\Pi'$ is approximately equal to $-16\eta^2/E^2$,

$$\left.\begin{aligned}\delta a_3/a &= -\omega^2\rho a^2\eta/(3E),\\ \delta a_1/\delta a_3 &= \text{expression of order } E/E' \text{ and so negligible}\end{aligned}\right\} \quad \text{.....(48)}.$$

If we suppose both (39) and (40) to hold, or the material to be absolutely incompressible, we find

$$\left.\begin{aligned}\delta a_1/a &= \frac{\omega^2\rho a^2}{2E\Pi'}\left(\frac{4}{EE'}-\frac{3}{4E^2}+\frac{1}{nE'}\right),\\ \delta a_3/a &= -\frac{\omega^2\rho a^2}{3E\Pi'}\left(\frac{10}{EE'}-\frac{7}{4E^2}+\frac{4}{nE'}-\frac{1}{2nE}\right)\end{aligned}\right\} \quad \text{.....(49)},$$

where $\Pi'$ is given by (43).

In the case of rotation, unlike that of gravitation, a slight departure from incompressibility has very little effect; we may thus regard the results (49) as close approximations when the material is *slightly* compressible. In particular, if the material, though absolutely incompressible under uniform pressure, is slightly compressible under other circumstances, we find under combined gravitation and rotation from (44) and (49)

$$\left.\begin{aligned}\delta a_1/a &= \frac{g\rho ap+\omega^2\rho a^2}{2E\Pi'}\left(\frac{4}{EE'}-\frac{3}{4E^2}+\frac{1}{nE'}\right),\\ \delta a_3/a &= -\frac{g\rho ap+\omega^2\rho a^2}{3E\Pi'}\left(\frac{10}{EE'}-\frac{7}{4E^2}+\frac{4}{nE'}-\frac{1}{2nE}\right)\end{aligned}\right\} \quad \text{.....(50)}.$$

So far as changes in the lengths of diameters are concerned, the gravitational and rotational influences are thus exactly parallel.

In the actual earth we have approximately

$$g\rho a/\omega^2\rho a^2 = 289,$$

and in such a case the gravitational and rotational effects on the diameters would be exactly equal if

$$p = 1/289 = \cdot 00346 \text{ approx.},$$

or

$$\eta = \cdot 4983 \text{ approx.}$$

If it were possible for $\eta$ to equal ·5017, the gravitational and rotational effects would in this case neutralise one another.

## SECTION III.

### Flat Ellipsoid.

§ 16. The next case considered is that of a very flat ellipsoid of material symmetrical with respect to the coordinate planes.

Supposing the axis of $z$ taken along the short diameter $2c$, and retaining only terms independent of $c$, we have as first approximations, with our previous notation,

$$\left.\begin{aligned}
&a_{11} = 3b^4/E_2, \quad a_{22} = 3a^4/E_1, \quad a_{33} = 3(a^4/E_1) + 3(b^4/E_2) + a^2b^2\{(1/n_3) - 2(\eta_{12}/E_1)\},\\
&a_{12} = -3a^2b^2\eta_{12}/E_1, \quad a_{13} = (b^4/E_2) - a^2b^2\eta_{12}/E_1, \quad a_{23} = a^4/E_1 - a^2b^2\eta_{12}/E_1,\\
&\Pi_{11} = \frac{8a^8}{E_1^2} + \frac{a^4b^4}{E_1E_2}(9 - \eta_{12}\eta_{21}) + \frac{a^6b^2}{E_1}\left(\frac{3}{n_3} - \frac{4\eta_{12}}{E_1}\right),\\
&\Pi_{22} = \frac{8b^8}{E_2^2} + \frac{a^4b^4}{E_1E_2}(9 - \eta_{12}\eta_{21}) + \frac{a^2b^6}{E_2}\left(\frac{3}{n_3} - \frac{4\eta_{21}}{E_2}\right),\\
&\Pi_{33} = 9a^4b^4(1 - \eta_{12}\eta_{21})/E_1E_2,\\
&\Pi_{12} = \frac{a^4b^4}{E_1}\left(\frac{1 - 5\eta_{12}\eta_{21}}{E_2} + \frac{3\eta_{12}}{n_3}\right) + \frac{8a^2b^2}{E_1E_2}(\eta_{21}a^4 + \eta_{12}b^4),\\
&\Pi_{13} = \Pi_{23} = -3a^4b^4(1 - \eta_{12}\eta_{21})/E_1E_2,\\
&\Pi = 3a^4b^4\frac{1 - \eta_{12}\eta_{21}}{E_1E_2}\left\{\frac{8a^4}{E_1} + \frac{8b^4}{E_2} + a^2b^2\left(\frac{3}{n_3} - \frac{4\eta_{12}}{E_1}\right)\right\}
\end{aligned}\right\} \quad \ldots\ldots\ldots (51).$$

Assuming

$$1 > \eta_{12}\eta_{21},$$

which can hardly fail to be universally true, $\Pi$ is essentially positive.

### Rotation about Short Axis.

§ 17. When the flat ellipsoid rotates with uniform angular velocity $\omega$ about its short axis, $P = Q = \omega^2$, $R = 0$, and the equations (16) take the form

$$a_{11}L + a_{12}M + a_{13}N = \tfrac{1}{2}\omega^2\rho b^2\{a^2(\eta_{12}/E_1) - b^2/E_2\},$$

$$a_{12}L + a_{22}M + a_{23}N = \tfrac{1}{2}\omega^2\rho a^2\{-a^2/E_1 + b^2(\eta_{21}/E_2)\},$$

$$a_{13}L + a_{23}M + a_{33}N = -\tfrac{1}{2}\omega^2\rho\left\{\frac{1}{E_1}a^2(a^2 - \eta_{12}b^2) + \frac{1}{E_2}b^2(b^2 - \eta_{21}a^2)\right\},$$

where $a_{11}$, &c. are given by (51).

Referring to the values found above for the determinant $\Pi$ and its minors, we find on reduction, remembering (2),

$$\left.\begin{aligned} L = M &= -\frac{1}{2}\frac{\omega^2\rho}{\Pi'}\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right), \\ N &= -\frac{\omega^2\rho}{\Pi'}\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-\frac{2a^2b^2\eta_{12}}{E_1}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (52);$$

where

$$\Pi' = \frac{8a^4}{E_1}+\frac{8b^4}{E_2}+a^2b^2\left(\frac{3}{n_3}-\frac{4\eta_{12}}{E_1}\right) \ldots\ldots\ldots\ldots (53).$$

We may reasonably regard $L$, $M$ and $N$ as essentially negative.

In our subsequent work the following result will be found useful,

$$3(L+M)+2N \equiv 2(3L+N) \equiv 2(3M+N) = -\omega^2\rho \ldots\ldots\ldots\ldots (54).$$

Putting in (13)

$$S = T = U = R = 0, \quad P = Q = \omega^2,$$

we have

$$\widehat{xx}/a^2 = (\tfrac{1}{2}\omega^2\rho + M + N)(1 - x^2/a^2 - y^2/b^2) - (\tfrac{1}{2}\omega^2\rho + 3M + N)z^2/c^2 - 2Ny^2/b^2.$$

Having regard to (54) we see that the coefficient of $z^2/c^2$ vanishes, and deduce

$$\widehat{xx} = -2a^2L(1 - x^2/a^2 - y^2/b^2) - 2Ny^2a^2/b^2,$$

where $L$ and $N$ are given by (52) and (53).
Similarly we find

$$\widehat{zz} = 2Lc^2(1 - 2x^2/a^2 - 2y^2/b^2 - z^2/c^2).$$

But we have been treating terms of order $c^2$ as negligible and so may regard $\widehat{zz}$ as vanishing.

Again we have

$$\widehat{yz} = 2Lyz, \quad \widehat{zx} = 2Mzx;$$

or these two shearing stresses are of order $c$, and so though they are large compared to $\widehat{zz}$ we may neglect them for a first approximation. The complete stress system remaining may be written

$$\left.\begin{aligned} \widehat{xx} &= -2a^2L(1 - x^2/a^2 - y^2/b^2) - 2Na^2b^2y^2/b^4, \\ \widehat{yy} &= -2b^2L(1 - x^2/a^2 - y^2/b^2) - 2Na^2b^2x^2/a^4, \\ \widehat{xy} &= 2Nxy \end{aligned}\right\} \ldots\ldots\ldots\ldots (55),$$

with $L$ and $N$ given as above by (52) and (53).

§ 18. If $\psi$ be the inclination to the $x$-axis of one of the principal stress axes in planes parallel to $xy$, we have

$$\begin{aligned} \cot 2\psi &= \tfrac{1}{2}(\widehat{xx} - \widehat{yy})/\widehat{xy} \\ &= \cot 2\phi - (a^2 - b^2)L(1 - x^2/a^2 - y^2/b^2) \div (2Nxy) \ldots\ldots\ldots\ldots (56); \end{aligned}$$

where $\phi$ is the inclination to the $x$-axis of the normal to the confocal

$$x^2/a^2 + y^2/b^2 = \lambda,$$

which passes through the point $x$, $y$, $z$.

This gives very readily the angles $\psi$ and $\frac{\pi}{2}+\psi$ made with the $x$-axis by the two principal stress axes which lie in the plane parallel to $xy$. The third principal stress axis is always parallel to the $z$-axis.

Without any reference to the values of $L$ and $N$ we see from (56) that

$$\psi = \phi,$$

for all values of $x$ and $y$ if $\qquad b^2 = a^2,$

and for all values of $b/a$ if $\qquad x^2/a^2 + y^2/b^2 = 1.$

We thus see that in a flat rotating spheroid, whatever be the relative values of the Young's moduli or Poisson's ratios, any perpendicular on the axis of rotation is a principal axis of stress at every point of its length.

Again for any shape of flat spheroid the principal stress axes at the rim in the central section $z=0$ coincide everywhere with the normal and tangent to the bounding ellipse.

The stress along the rim normal vanishes in accordance with the surface conditions, while the stress $\widehat{u}$ along the tangent is given by

$$\widehat{u} = p^2\{(y^2/b^4)\,\widehat{xx} + (x^2/a^4)\,\widehat{yy} - 2\,(xy/a^2b^2)\,\widehat{xy}\},$$

where $p$ is the perpendicular from the centre on the tangent at $x$, $y$.

Referring to (55), and remembering that

$$x^2/a^4 + y^2/b^4 = p^{-2},$$

we easily find

$$\widehat{u} = -\,2Na^2b^2/p^2 \qquad\qquad (57);$$

or, writing in its value for $N$,

$$\widehat{u} = 2\omega^2\rho\,(a^2b^2/p^2)\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-\frac{2a^2b^2\eta_{12}}{E_1}\right)\div\left\{\frac{8a^4}{E_1}+\frac{8b^4}{E_2}+a^2b^2\left(\frac{3}{n_3}-\frac{4\eta_{12}}{E_1}\right)\right\} \qquad (58).$$

The stress along the tangent to the rim in the central section is thus a traction, which varies inversely as the square of the perpendicular from the centre on the tangent.

§ 19. The strains which do not vanish are, as a first approximation,

$$\left.\begin{aligned}
s_x &= -\,2\,(L/E_1)\,(a^2-\eta_{12}b^2)\,(1-x^2/a^2-y^2/b^2) - 2\,(N/E_1)\,(a^2y^2/b^2-\eta_{12}b^2x^2/a^2),\\
s_y &= -\,2\,(L/E_2)\,(b^2-\eta_{21}a^2)\,(1-x^2/a^2-y^2/b^2) - 2\,(N/E_2)\,(b^2x^2/a^2-\eta_{21}a^2y^2/b^2),\\
s_z &= 2\,(L/E_3)(\eta_{31}a^2+\eta_{32}b^2)(1-x^2/a^2-y^2/b^2) + 2(N/E_3)(\eta_{31}a^2y^2/b^2+\eta_{32}b^2x^2/a^2),\\
\sigma_{xy} &= 2Nxy/n_3
\end{aligned}\right\} \qquad (59);$$

where $L$ and $N$ are given as before by (52) and (53).

To the present degree of approximation, the strains, like the stresses, do not vary with $z$; and at the rim in the central section $z=0$ they depend on the constant $N$ only.

Along the axis of rotation the strains are constants given by the simple expressions

$$\left.\begin{aligned} s_x &= -2(L/E_1)(a^2-\eta_{12}b^2), \\ s_y &= -2(L/E_2)(b^2-\eta_{21}a^2), \\ s_z &= 2(L/E_3)(\eta_{31}a^2+\eta_{32}b^2) \end{aligned}\right\} \quad \text{.....(60)};$$

where $L$, as shown by (52), is a negative quantity.

An $\eta$ in excess of 0·5 is at least highly exceptional, thus supposing $a$ to be the longer semi-axis we may regard $s_x$ at the axis as essentially positive, or a stretch. On the other hand $s_y$ at the axis is positive or negative according as

$$b/a > \text{or} < \sqrt{\eta_{21}}.$$

For the changes in the lengths of the semi-axes we find from (59), by integration and substitution for $L$ and $N$,

$$\left.\begin{aligned} \delta a/a &= \frac{2}{3}\frac{\omega^2\rho}{E_1\Pi'}\left[(a^2-\eta_{12}b^2)\left\{\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+a^2b^2\left(\frac{1}{n_3}-\frac{\eta_{12}}{E_1}\right)\right\}-(b^2-\eta_{21}a^2)\frac{\eta_{12}b^4}{E_2}\right], \\ \delta b/b &= \frac{2}{3}\frac{\omega^2\rho}{E_2\Pi'}\left[(b^2-\eta_{21}a^2)\left\{\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+a^2b^2\left(\frac{1}{n_3}-\frac{\eta_{12}}{E_1}\right)\right\}-(a^2-\eta_{12}b^2)\frac{\eta_{21}a^4}{E_1}\right], \\ \delta c/c &= -\frac{\omega^2\rho}{E_3\Pi'}(\eta_{31}a^2+\eta_{32}b^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right) \end{aligned}\right\} \quad \text{...(61)},$$

where $\Pi'$ is given by (53).

In passing, the following elegant relation may be noted

$$\frac{3(a^2\delta a/a-b^2\delta b/b)}{a^4/E_1-b^4/E_2}+\frac{2\delta c/c}{(\eta_{31}a^2+\eta_{32}b^2)/E_3}=0 \quad \text{.....(62)}.$$

Regarding $\Pi'$ as essentially positive, we see that $\delta c/c$ is invariably negative; or the short axis, about which the rotation occurs, necessarily shortens. The two perpendicular axes if similar in length in general both lengthen. If $b$, however, is much smaller than $a$ it will usually shorten.

For instance, if $\qquad b^2/a^2=\eta_{21}$ .....(63),

we have $\qquad \delta b/b=-\frac{2}{3}\omega^2\rho a^2(\eta_{12}/E_1)(1-\eta_{12}\eta_{21})\div(8+4\eta_{12}\eta_{21}+3\eta_{21}E_1/n_3)$.....(64).

The relations (60) and (57), it should be noticed, supply simple physical meanings to the constants $L$ and $N$ of the solution.

## SECTION IV.

### THIN ELLIPTIC DISK ROTATING ABOUT THE PERPENDICULAR TO ITS PLANE THROUGH THE CENTRE.

§ 20. In a previous paper* I have shown that in isotropic material the first approximations to the stresses and strains in a thin elliptic disk may be derived by applying the constant multiplier

$$\{4a^4 + (3+\eta)\,a^2b^2 + 4b^4\} \div (3a^4 + 2a^2b^2 + 3b^4)$$

to the values of the corresponding stresses and strains in a flat ellipsoid of the same (central) section $\pi ab$, and equal axial thickness $2c$ or $2l$. A similar result holds when the material is of the more general type dealt with in the present paper, the constant multiplier being alone different. To find the suitable constant multiplier we may proceed as follows:

The mean values of the stresses, as I showed in an earlier paper†, are given by simple formulae of the type

$$\iiint \widehat{xx}\,dxdydz = \iiint Xx\,dxdydz + \iint Fx\,dS \qquad (65);$$

where $X$ is the $x$-component of the bodily forces per unit volume, and $F$ the $x$-component of the surface forces per unit surface. The volume integrals extend throughout the entire volume, the surface integral over the whole surface of the solid.

In the present case we thus have

$$\iiint \widehat{xx}\,dxdydz = \omega^2\rho \iiint x^2\,dxdydz \qquad (66).$$

Supposing $C$ to denote the constant multiplier required for transformation from the flat ellipsoid to the thin disk, we find for the disk from (55)

$$\widehat{xx} = -2CLa^2(1 - x^2/a^2 - y^2/b^2) - 2CNa^2b^2y^2/b^4,$$

where $L$ and $N$ are given by (52) and (53).

Substituting for $\widehat{xx}$ in (66) and integrating, we find

$$-C\,.\,2\pi abc\,.\,a^2(L + \tfrac{1}{2}N) = \omega^2\rho\,.\,2\pi abc\,.\,a^2/4;$$

whence

$$C = -\omega^2\rho/(4L + 2N),$$
$$= (3L + N)/(2L + N) \text{ by (54).}$$

Referring to (52), we have at once

$$C = \left\{\frac{8a^4}{E_1} + \frac{8b^4}{E_2} + a^2b^2\left(\frac{3}{n_3} - \frac{4\eta_{12}}{E_1}\right)\right\} \div \left\{\frac{6a^4}{E_1} + \frac{6b^4}{E_2} + 2a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1}\right)\right\} \qquad (67).$$

Thus, writing for shortness

$$\frac{6a^4}{E_1} + \frac{6b^4}{E_2} + 2a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1}\right) = \Pi'' \qquad (68),$$

* (A), p. 49.

† *Camb. Phil. Trans.* Vol. xv. equation (109), p. 336.

we have as first approximations to the stresses and strains in the thin rotating elliptic disk :—

$$\left.\begin{aligned}
\widehat{xx} &= \frac{\omega^2\rho a^2}{\Pi''}\left[\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\frac{y^2}{b^2}\right],\\
\widehat{yy} &= \frac{\omega^2\rho b^2}{\Pi''}\left[\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\frac{x^2}{a^2}\right],\\
\widehat{xy} &= -\frac{2\omega^2\rho}{\Pi''}\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-\frac{2a^2b^2\eta_{12}}{E_1}\right)xy,\\
\widehat{zz} &= \widehat{xz} = \widehat{yz} = 0
\end{aligned}\right\}\quad\ldots\ldots(69);$$

$$\left.\begin{aligned}
s_x &= \frac{\omega^2\rho}{E_1\Pi''}\left[(a^2-\eta_{12}b^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right.\\
&\qquad\left.+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\left(\frac{a^2y^2}{b^2}-\eta_{12}\frac{b^2x^2}{a^2}\right)\right],\\
s_y &= \frac{\omega^2\rho}{E_2\Pi''}\left[(b^2-\eta_{21}a^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right.\\
&\qquad\left.+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\left(\frac{b^2x^2}{a^2}-\eta_{21}\frac{a^2y^2}{b^2}\right)\right],\\
s_z &= -\frac{\omega^2\rho}{E_3\Pi''}\left[(\eta_{31}a^2+\eta_{32}b^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right.\\
&\qquad\left.+2\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)\left(\eta_{31}\frac{a^2y^2}{b^2}+\eta_{32}\frac{b^2x^2}{a^2}\right)\right],\\
\sigma_{xy} &= -\frac{2\omega^2\rho}{n_3\Pi''}\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-2a^2b^2\frac{\eta_{12}}{E_1}\right)xy,\\
\sigma_{xz} &= \sigma_{yz} = 0
\end{aligned}\right\}\quad\ldots\ldots(70).$$

§ 21. The position of the principal stress axes in the disk is given by the same equation (56) as applies to the flat ellipsoid. Again, over the perimeter of the disk the normal stress component vanishes and the tangential component, in the plane of the cross-section, is given, cf. (58), by

$$\widehat{u} = 2\omega^2\rho(a^2b^2/p^2)\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-\frac{2a^2b^2\eta_{12}}{E_1}\right)\div\left\{\frac{6a^4}{E_1}+\frac{6b^4}{E_2}+2a^2b^2\left(\frac{1}{n_3}-\frac{2\eta_{12}}{E_1}\right)\right\}\quad\ldots\ldots\ldots\ldots(71),$$

where $p$ is the perpendicular on the tangent from the centre of the ellipse.

The increments $\delta a$, $\delta b$ and $\delta l$ $(l=c)$ in the semi-axes of the ellipse and the axial semi-thickness are given by (61) when $\Pi'$ is replaced by $\Pi''$. The relation (62) applies equally to the disk.

Employing $\delta l$ as above, we find from the value of $s_z$ that the displacement $\gamma$ parallel to the axis of rotation is given by

$$\gamma = (\delta l/l)\,z\,(1-x^2/a_1^2-y^2/b_1^2)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(72);$$

where

$$\left.\begin{aligned} a_1^2 &\equiv (\eta_{31}a^2+\eta_{32}b^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\div\left\{\eta_{31}\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)+\eta_{32}\left(\frac{1}{n_3}+\frac{4\eta_{12}}{E_1}\right)b^4\right\}, \\ b_1^2 &= (\eta_{31}a^2+\eta_{32}b^2)\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\div\left\{\eta_{31}\left(\frac{1}{n_3}+\frac{4\eta_{12}}{E_1}\right)a^4+\eta_{32}\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\right\} \end{aligned}\right\} \quad \ldots(73).$$

Since $\delta l$ is negative, assuming $\Pi''$ positive, (72) shows very clearly how the originally plane cross-sections parallel to the faces of the disk become paraboloids whose concavities are directed away from the central section, and whose curvature increases with the distance $z$ from that section.

The curvature at the centre of an originally plane section is greatest in the $zy$ or in the $zx$ plane according as

$$a_1^2 > \text{ or } < b_1^2.$$

The value of $a_1^2/b_1^2$ depends on the shape of the section as well as on the elastic properties of the medium. Assuming as before $a > b$, we easily find $a_1^2 > b_1^2$ if either

$$\eta_{32}/\eta_{31} = 1,$$

or

$$(a/b)^4 > \eta_{32}/\eta_{31} > 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74).$$

Thus the curvature is greatest in the plane containing the shorter axis of the ellipse if $\eta_{32}$ and $\eta_{31}$ are equal or if (74) holds.

Whilst the reduction in the thickness of the disk diminishes as we retire from the axis of rotation it remains a reduction right up to the rim. For it is obvious from (70) that $\gamma_s$, the value of $\gamma$ over the curved surface, is given by

$$\gamma_s = -z\,(2\omega^2\rho/E_3\Pi'')\left(\frac{a^4}{E_1}+\frac{b^4}{E_2}-\frac{2a^2b^2\eta_{12}}{E_1}\right)\left(\eta_{31}\frac{a^2y^2}{b^2}+\eta_{32}\frac{b^2x^2}{a^2}\right) \quad \ldots\ldots\ldots\ldots\ldots (75),$$

and $\dfrac{a^4}{E_1}+\dfrac{b^4}{E_2}-\dfrac{2a^2b^2\eta_{12}}{E_1}$ can hardly fail to be positive.

It may be worth noticing that the reduction in the rim thickness is greatest at the ends of the minor axis or at the ends of the major axis according as

$$a^2\eta_{31} > \text{ or } < b^2\eta_{32}.$$

## SECTION V.

### THIN ROTATING CIRCULAR DISK OF MATERIAL SYMMETRICAL ROUND THE AXIS.

§ 22. When the disk is complete the approximate solution can be deduced from the results of the previous section by putting

$$b = a, \quad E_2 = E_1 = E', \quad \&c.$$

To include, however, the case of an annular disk, we must make a fresh start. In obtaining the following results I freely availed myself of a previous solution*, with

* 'On thin rotating isotropic disks,' *Camb. Phil. Soc. Proc.* Vol. VII. pp. 201—215, 1891.

which I foresaw that the present solution would agree in type. It will suffice to indicate generally the method of procedure.

Representing by $2l$ the thickness of the disk, by $a$ and $a'$ the radii of its outer and inner cylindrical boundaries, and employing cylindrical coordinates $r$, $\phi$ $z$, let us assume

$$\left.\begin{aligned} \widehat{rr} &= A(a^2-r^2)(1-a'^2/r^2)+B(l^2-3z^2),\\ \widehat{\phi\phi} &= C+Dr^2+Fr^{-2}+B(l^2-3z^2),\\ \widehat{zz} &= \widehat{r\phi}=\widehat{rz}=\widehat{\phi z}=0 \end{aligned}\right\} \quad \text{.....} \quad (76).$$

From the body-stress equation

$$\frac{d\widehat{rr}}{dr}+\frac{1}{r}(\widehat{rr}-\widehat{\phi\phi})+\omega^2\rho r=0$$

we find

$$C=A(a^2+a'^2),\quad D=\omega^2\rho-3A,\quad F=Aa^2a'^2.$$

Only two of the equations of compatibility are not identically satisfied, and these two give us

$$A=\omega^2\rho(3+\eta')/8,$$
$$B=\omega^2\rho\eta(1+\eta')E'\div\{6(1-\eta')E\}.$$

Substituting in (76) the values thus found for the constants, we have

$$\left.\begin{aligned} \widehat{rr} &= \frac{1}{8}\omega^2\rho(3+\eta')(a^2-r^2)\left(1-\frac{a'^2}{r^2}\right)+\frac{1}{6}\frac{\omega^2\rho\eta(1+\eta')E'}{(1-\eta')E}(l^2-3z^2),\\ \widehat{\phi\phi} &= \frac{1}{8}\omega^2\rho\left\{(3+\eta')\left(a^2+a'^2+\frac{a^2a'^2}{r^2}\right)-(1+3\eta')r^2\right\}+\frac{1}{6}\frac{\omega^2\rho\eta(1+\eta')E'}{(1-\eta')E}(l^2-3z^2),\\ \widehat{zz} &= \widehat{r\phi}=\widehat{rz}=\widehat{\phi z}=0 \end{aligned}\right\} \quad \text{.....} \quad (77).$$

The displacements ($u$ along $r$, and $w$ parallel to $z$) are easily obtained from the relations

$$u/r\equiv s_\phi=(\widehat{\phi\phi}-\eta'\widehat{rr})/E',$$
$$\frac{dw}{dz}\equiv s_z=-\eta(\widehat{rr}+\widehat{\phi\phi})/E;$$

whence we have, as $w$ must vanish with $z$ by symmetry,

$$\left.\begin{aligned} u &= \frac{1}{8}\frac{\omega^2\rho}{E'}\left\{(1-\eta')(3+\eta')r(a^2+a'^2)-(1-\eta'^2)r^3+(1+\eta')(3+\eta')\frac{a^2a'^2}{r}\right\}\\ &\qquad\qquad +\frac{1}{6}\frac{\omega^2\rho\eta(1+\eta')}{E}r(l^2-3z^2),\\ w &= -\frac{1}{4}\frac{\omega^2\rho\eta z}{E}\{(3+\eta')(a^2+a'^2)-2(1+\eta')r^2\}-\frac{1}{3}\frac{\omega^2\rho\eta^2(1+\eta')E'}{(1-\eta')E^2}z(l^2-z^2) \end{aligned}\right\} \quad \text{.....} \quad (78).$$

§ 23. If in (77) and (78) we put $a'=0$ we obtain the correct values of the stresses and displacements in a complete disk of radius $a$.

It should be noticed, however, that the strains and stresses near the inner surface of a *nearly* complete disk (*i.e.* one in which $a'/a$ is very small) are totally different

from those at the same axial distance in a complete disk. This is due to the fact that even supposing $(a'/a)^2$ negligible, $(a'/r)^2$ approaches unity as $r$ approaches $a'$.

The results supplied by (77) and (78) for a complete disk differ from those we should deduce from the formulae of Section IV., but only through containing the terms in $l^2 - 3z^2$ and $z(l^2 - z^2)$. Now we neglected terms of this order in Sections III. and IV., where terms in $(c/a)^2$—*i.e.* terms of order $(l/a)^2$—were omitted.

It is obvious in (77) and (78) that the mean values of $\widehat{rr}$, $\widehat{\phi\phi}$ and $u$ taken between $z = -l$ and $z = +l$, or through the thickness of the disk, are unaffected by the presence of the terms in $l^2$ and $z^2$, and the same is true of the values of $w$ over the faces of the disk.

§ 24. It may be well at this stage to make the status of our solution for the circular disk perfectly clear. It is not in general a *complete* solution of the mathematical equations. It satisfies indeed all the internal elastic solid equations, and all but one of the surface equations; but instead of making $\widehat{rr}$ identically zero at *every* point of both cylindrical surfaces, it only gives

$$\int_{-l}^{+l} \widehat{rr}\, dz = 0$$

over these surfaces. In other words, it only makes the statical *resultant* of the radial forces along a generator vanish. The solution is thus based on what Prof. Pearson terms the theory of equipollent systems of loading. According to the theory, which is very generally if not universally accepted, the error in such a solution is insensible, except in the immediate vicinity of the surfaces of small thickness—here the circular rims—where there is failure to satisfy the exact surface conditions. As the rim values of $\widehat{rr}$ in the present case are only of the order $l^2$ of small quantities, our solution is presumably an exceptionally favourable specimen of its class. Still it would not be legitimate to apply it without further investigation to the species of anchor ring which arises when $a - a'$ is comparable with $l$.

At first sight, it might appear better to have omitted the terms in $l^2$ and $z^2$ altogether; because in their absence $\widehat{rr}$ would vanish exactly over both rims. If, however, we omitted those terms, we should be unable to satisfy all the internal equations. Such a failure, in the absence of special knowledge, is much more serious than failure to satisfy a surface condition. For in dealing with internal equations we get, through differentiating, contributions of like magnitude from terms that are of widely different importance in the displacements and stresses. It is thus almost impossible to judge whether failure to satisfy an internal equation is trivial or absolutely fatal.

In the present case, while the terms in $l^2$ and $z^2$ serve mainly to save the proprieties and silence criticism, they fulfil a useful purpose in indicating the degree of approximation reached and the circumstances modifying it. For instance, the solution

becomes absolutely exact if

$$\eta = 0, \text{ or } E'/E = 0;$$

and it is the more exact the smaller $\eta$ or $E'/E$ is.

On the other hand if $E'/E$ be large the solution has a very limited application.

§ 25. When $E$ and $E'$ are of the same order of magnitude we may omit the terms in $l^2$ and $z^2$ in ordinary practical applications. When these terms are omitted I shall use the notation $\widehat{\overline{rr}}$, $\bar{u}$, &c. When the material is isotropic the values of $\bar{u}$, $\bar{w}$, &c., constitute what I have called elsewhere* the 'Maxwell solution,' as being the solution to which Maxwell's treatment of the problem would have led him in 1853 but for some small inaccuracies in his work.

It is noteworthy that $\widehat{\overline{rr}}$ and $\widehat{\overline{\phi\phi}}$ depend on no elastic constant other than $\eta'$, while $\bar{u}$ is independent of $\eta$ or $E$. Thus the stresses and radial displacement are exactly the same as in an isotropic material whose Young's modulus in $E'$ and Poisson's ratio $\eta'$.

The longitudinal displacement $\bar{w}$ on the other hand depends on $\eta$ and $E$, but even in its case the law of variation with the axial distance depends only on $\eta'$.

For the increments in the radii $a$ and $a'$, and in the semi-thickness at the two rims, we find

$$\left.\begin{aligned}(\overline{\delta a}/a) &= (\omega^2\rho/4E')\{(1-\eta')a^2+(3+\eta')a'^2\},\\ (\overline{\delta a'}/a') &= (\omega^2\rho/4E')\{(3+\eta')a^2+(1-\eta')a'^2\}\end{aligned}\right\}\quad\text{.....(79)};$$

$$\left.\begin{aligned}(\delta l/l)_{r=a} &= -(\omega^2\rho\eta/4E)\{(1-\eta')a^2+(3+\eta')a'^2\},\\ (\delta l/l)_{r=a'} &= -(\omega^2\rho\eta/4E)\{(3+\eta')a^2+(1-\eta')a'^2\}\end{aligned}\right\}\quad\text{.....(80)}.$$

From these we deduce the following elegant relations

$$\left.\begin{aligned}(\delta l/l)_{r=a} &= -\eta(E'/E)(\overline{\delta a}/a) = -\eta''(\overline{\delta a}/a),\\ (\delta l/l)_{r=a'} &= -\eta(E'/E)(\overline{\delta a'}/a') = -\eta''(\overline{\delta a'}/a'),\\ (\delta l/l)_{r=a}+(\delta l/l)_{r=a'} &= -\omega^2\rho(\eta/E)(a^2+a'^2)\end{aligned}\right\}\quad\text{.....(81)}.$$

The arithmetic mean of the reductions in the thickness at the two rims of the disk is thus independent of $\eta'$ or $E'$. The reduction in thickness is invariably greatest at the inner rim.

Originally plane sections parallel to the faces lie during rotation on paraboloids of revolution, the radius of curvature at whose vertices equals $E \div \{\omega^2\rho\eta(1+\eta')z\}$.

The curvature increases as we approach the faces $z = \pm l$. The general character of the phenomena is the same as when the material is isotropic (see *Camb. Phil. Soc. Proc.* Vol. VII. pp. 201—215).

* *Camb. Phil. Soc. Proc.* Vol. VII. p. 209.

## SECTION VI.

### ELONGATED ELLIPSOID, $c/a$ AND $c/b$ VERY LARGE.

§ 26. Retaining only the highest power of $c$ in each case, we have for three-plane symmetry

$$\left.\begin{aligned}
&a_{11} = 3c^4/E_3, \quad a_{22} = 3c^4/E_3, \quad a_{33} = 3\,(a^4/E_1) + 3\,(b^4/E_2) + a^2b^2\,\{1/n_3 - 2\eta_{12}/E_1\},\\
&a_{12} = c^4/E_3, \quad a_{13} = -\,c^2\,(3a^2\eta_{31}/E_3 + b^2\eta_{32}/E_3), \quad a_{23} = -\,c^2\,(a^2\eta_{31} + 3b^2\eta_{32})/E_3,\\
&\Pi_{11} = (c^4/E_3)\left[\frac{a^4}{E_1}(9 - \eta_{13}\eta_{31}) + \frac{9b^4}{E_2}(1 - \eta_{23}\eta_{32}) + 3a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1} - \frac{2\eta_{13}\eta_{32}}{E_1}\right)\right],\\
&\Pi_{22} = (c^4/E_3)\left[\frac{9a^4}{E_1}(1 - \eta_{13}\eta_{31}) + \frac{b^4}{E_2}(9 - \eta_{23}\eta_{32}) + 3a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{21}}{E_2} - \frac{2\eta_{23}\eta_{31}}{E_2}\right)\right],\\
&\Pi_{33} = 8c^8/E_3^2,\\
&\Pi_{12} = (c^4/E_3)\left[-\frac{3a^4}{E_1}(1 - \eta_{13}\eta_{31}) - \frac{3b^4}{E_2}(1 - \eta_{23}\eta_{32}) + a^2b^2\left(\frac{10\eta_{31}\eta_{32}}{E_3} + \frac{2\eta_{12}}{E_1} - \frac{1}{n_3}\right)\right],\\
&\Pi_{13} = 8c^6a^2\eta_{13}/(E_1E_3),\\
&\Pi_{23} = 8c^6b^2\eta_{23}/(E_2E_3),\\
&\Pi = 8\,(c^8/E_3^2)\left[\frac{3a^4}{E_1}(1 - \eta_{13}\eta_{31}) + \frac{3b^4}{E_2}(1 - \eta_{23}\eta_{32}) + a^2b^2\left(\frac{1}{n_3} - \frac{2\eta_{12}}{E_1} - \frac{2\eta_{13}\eta_{32}}{E_1}\right)\right]
\end{aligned}\right\} \quad \ldots\ldots(82).$$

### ROTATION ABOUT THE LONG AXIS $2c$.

§ 27. The values of the $\varpi$'s in equations (16) are as follows:

$$\left.\begin{aligned}
&\varpi_1 = \varpi_2 = \tfrac{1}{2}\omega^2\rho c^2\,(a^2\eta_{31} + b^2\eta_{32})/E_3,\\
&\varpi_3 = \tfrac{1}{2}\omega^2\rho\left\{\frac{a^2}{E_1}(\eta_{13}b^2 - a^2) + \frac{b^2}{E_2}(\eta_{21}a^2 - b^2)\right\}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(83).$$

Substituting the above values of the $\Pi$'s and $\varpi$'s in the equations (18) we have the values of the constants $L$, $M$, $N$ of the general solution. Thus for $N$ we find

$$N = \frac{\tfrac{1}{2}\omega^2\rho\left\{-\dfrac{a^4}{E_1}(1 - \eta_{13}\eta_{31}) - \dfrac{b^4}{E_2}(1 - \eta_{23}\eta_{32}) + \dfrac{2a^2b^2}{E_1}(\eta_{12} + \eta_{13}\eta_{32})\right\}}{\dfrac{3a^4}{E_1}(1 - \eta_{13}\eta_{31}) + \dfrac{3b^4}{E_2}(1 - \eta_{23}\eta_{32}) + a^2b^2\left(\dfrac{1}{n_3} - \dfrac{2\eta_{12}}{E_1} - \dfrac{2\eta_{13}\eta_{32}}{E_1}\right)} \quad \ldots\ldots\ldots\ldots(84).$$

It is unnecessary to record the values of $L$ and $M$ as I have eliminated these quantities by aid of the following relations, which are not very difficult to verify:—

$$\left.\begin{aligned}
&c^2\,(L - M) = (\eta_{31}a^2 - \eta_{32}b^2)\,N,\\
&c^2\,(L + M) = (\eta_{31}a^2 + \eta_{32}b^2)\,(N + \tfrac{1}{4}\omega^2\rho)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(85);$$

whence

$$\left.\begin{aligned}
&c^2L = \eta_{31}a^2N + \tfrac{1}{8}\omega^2\rho\,(\eta_{31}a^2 + \eta_{32}b^2),\\
&c^2M = \eta_{32}b^2N + \tfrac{1}{8}\omega^2\rho\,(\eta_{31}a^2 + \eta_{32}b^2)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(86).$$

Retaining $N$ (given explicitly by (84)) for brevity in the expressions for the stresses and strains, we have

$$\left.\begin{aligned}
\widehat{xx} &= a^2\left[(\tfrac{1}{2}\omega^2\rho+N)\left(1-\frac{x^2}{a^2}-\frac{z^2}{c^2}\right)-(\tfrac{1}{2}\omega^2\rho+3N)\frac{y^2}{b^2}\right],\\
\widehat{yy} &= b^2\left[-(\tfrac{1}{2}\omega^2\rho+3N)\frac{x^2}{a^2}+(\tfrac{1}{2}\omega^2\rho+N)\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)\right],\\
\widehat{zz} &= (\eta_{31}a^2+\eta_{32}b^2)(\tfrac{1}{2}\omega^2\rho+N)\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}-\frac{z^2}{c^2}\right)+(\eta_{31}a^2-\eta_{32}b^2)N\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right),\\
\widehat{xy} &= 2Nxy,\\
&\widehat{yz} \text{ and } \widehat{zx} \text{ of order } a/c \text{ or } b/c, \text{ and so negligible}
\end{aligned}\right\}\quad\ldots\ldots(87);$$

$$\left.\begin{aligned}
s_x &= \tfrac{1}{2}(\omega^2\rho/E_1)\left\{(a^2-\eta_{12}b^2)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)-\tfrac{1}{2}\eta_{13}(\eta_{31}a^2+\eta_{32}b^2)\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}-\frac{z^2}{c^2}\right)\right\}\\
&\quad+(N/E_1)\left\{a^2(1-\eta_{13}\eta_{31})\left(1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}-\frac{z^2}{c^2}\right)-b^2(\eta_{12}+\eta_{13}\eta_{32})\left(1-\frac{3x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)\right\},\\
s_y &= \tfrac{1}{2}(\omega^2\rho/E_2)\left\{(b^2-\eta_{21}a^2)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)-\tfrac{1}{2}\eta_{23}(\eta_{31}a^2+\eta_{32}b^2)\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}-\frac{z^2}{c^2}\right)\right\}\\
&\quad+(N/E_2)\left\{b^2(1-\eta_{23}\eta_{32})\left(1-\frac{3x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)-a^2(\eta_{21}+\eta_{23}\eta_{31})\left(1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}-\frac{z^2}{c^2}\right)\right\},\\
s_z &= -\tfrac{1}{4}\omega^2\rho(\eta_{31}a^2+\eta_{32}b^2)(1-z^2/c^2)/E_3,\\
\sigma_{xy} &= 2Nxy/n_3,\\
&\sigma_{yz} \text{ and } \sigma_{zx} \text{ of order } a/c \text{ or } b/c \text{ and so negligible}
\end{aligned}\right\}\ldots(88).$$

§ 28. For the displacement parallel to the long axis we have

$$\gamma = -\tfrac{1}{4}\omega^2\rho(\eta_{31}a^2+\eta_{32}b^2)\,z\left(1-\frac{1}{3}\frac{z^2}{c^2}\right)\Big/E_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(89).$$

Sections perpendicular to the axis of rotation thus remain plane. The shortening of the long semi-axis is given by

$$\delta c/c = -\tfrac{1}{6}\omega^2\rho(\eta_{31}a^2+\eta_{32}b^2)/E_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(90).$$

Using undashed letters as immediately above for the case of the long ellipsoid, and dashed letters for the case of the flat ellipsoid of Section III., the velocity of rotation, the material and the axes $2a$, $2b$ being the same in the two cases, we find from (90), (61) and (53)

$$(\delta c/c)\div(\delta c'/c') = 1-\tfrac{1}{6}\left\{\frac{4a^4}{E_1}+\frac{4b^4}{E_2}+a^2b^2\left(\frac{3}{n_3}+\frac{4\eta_{12}}{E_1}\right)\right\}\div\left(\frac{2a^4}{E_1}+\frac{2b^4}{E_2}+\frac{a^2b^2}{n_3}\right)\ldots\ldots\ldots(91).$$

Thus the shortening per unit of length in the axis of rotation is less in the elongated than in the flat ellipsoid.

For the increments in the other principal semi-axes of the elongated ellipsoid we have

$$\left.\begin{aligned}\delta a/a &= \frac{1}{3}\frac{\omega^2\rho}{E_1}\{a^2-\eta_{12}b^2-\tfrac{1}{4}\eta_{13}(\eta_{31}a^2+\eta_{32}b^2)\}+\frac{2}{3}\frac{N}{E_1}a^2(1-\eta_{12}\eta_{21}),\\ \delta b/b &= \frac{1}{3}\frac{\omega^2\rho}{E_2}\{b^2-\eta_{21}a^2-\tfrac{1}{4}\eta_{23}(\eta_{31}a^2+\eta_{32}b^2)\}+\frac{2}{3}\frac{N}{E_2}b^2(1-\eta_{23}\eta_{32})\end{aligned}\right\}\text{.....(92).}$$

The values obtained for $\delta a/a$ and $\delta b/b$ when its value (84) is substituted for $N$ are somewhat lengthy even for a spheroid ($b=a$).

As, however, the influence of the elastic structure is very clearly exhibited in the case of a spheroid, I shall record the value obtained for the *difference* in the expansions of the two semi-axes taken along the directions of the two principal moduli $E_1$ and $E_2$. It is given by

$$\begin{aligned}\frac{\delta a_1-\delta a_2}{\omega^2\rho a^2}&\left\{\frac{3}{E_1}(1-\eta_{13}\eta_{31})+\frac{3}{E_2}(1-\eta_{23}\eta_{32})+\frac{1}{n_3}-\frac{2\eta_{12}}{E_1}-\frac{2\eta_{13}\eta_{32}}{E_1}\right\}\\ &=\frac{2}{3}\left(\frac{1}{E_1}-\frac{1}{E_2}\right)\left\{\frac{1}{E_1}(1-\eta_{13}\eta_{31})+\frac{1}{E_2}(1-\eta_{23}\eta_{32})+\frac{1}{2n_3}\right\}\\ &\quad+\frac{1}{12}\cdot\frac{1}{E_3}(\eta_{31}^2-\eta_{32}^2)\left\{\frac{1}{E_1}(1-\eta_{13}\eta_{31})+\frac{1}{E_2}(1-\eta_{23}\eta_{32})-\frac{1}{n_3}-\frac{6\eta_{12}}{E_1}-\frac{6\eta_{12}\eta_{32}}{E_1}\right\}\text{.....(93).}\end{aligned}$$

By supposing equality first between $E_1$ and $E_2$, and secondly between $\eta_{31}$ and $\eta_{32}$, we readily see how $\delta a_1-\delta a_2$ depends on the difference between elastic moduli and on the difference between Poisson's ratios.

## SECTION VII.

### LONG ELLIPTIC CYLINDER ROTATING ABOUT ITS LONG AXIS.

§ 29. By a *long* elliptic cylinder is meant one whose length $2l$ is very large compared with the diameters $2a$, $2b$ of the cross-section. The solution for the elliptic cylinder—terms of order $a/l$ or $b/l$ being neglected—is obtained from that for the elongated ellipsoid by simply omitting all the terms in $z^2$. We thus have

$$\left.\begin{aligned}\widehat{xx} &= a^2\left[(\tfrac{1}{2}\omega^2\rho+N)\left(1-\frac{x^2}{a^2}\right)-(\tfrac{1}{2}\omega^2\rho+3N)\frac{y^2}{b^2}\right],\\ \widehat{yy} &= b^2\left[-(\tfrac{1}{2}\omega^2\rho+3N)\frac{x^2}{a^2}+(\tfrac{1}{2}\omega^2\rho+N)\left(1-\frac{y^2}{b^2}\right)\right],\\ \widehat{zz} &= (\eta_{31}a^2+\eta_{32}b^2)(\tfrac{1}{4}\omega^2\rho+N)\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}\right)+(\eta_{31}a^2-\eta_{32}b^2)N\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right),\\ \widehat{xy} &= 2Nxy,\\ \widehat{xz} &= \widehat{yz}=0\end{aligned}\right\}\text{.....(94);}$$

$$\left.\begin{aligned}
s_x &= \frac{1}{2}\frac{\omega^2\rho}{E_1}\left\{(a^2-\eta_{12}b^2)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\tfrac{1}{2}\eta_{12}(\eta_{21}a^2+\eta_{22}b^2)\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}\right)\right\}\\
&\quad+\frac{N}{E_1}\left\{a^2(1-\eta_{12}\eta_{21})\left(1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}\right)-b^2(\eta_{12}+\eta_{12}\eta_{22})\left(1-\frac{3x^2}{a^2}-\frac{y^2}{b^2}\right)\right\},\\
s_y &= \frac{1}{2}\frac{\omega^2\rho}{E_2}\left\{(b^2-\eta_{21}a^2)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\tfrac{1}{2}\eta_{22}(\eta_{21}a^2+\eta_{22}b^2)\left(1-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}\right)\right\}\\
&\quad+\frac{N}{E_2}\left\{b^2(1-\eta_{22}\eta_{22})\left(1-\frac{3x^2}{a^2}-\frac{y^2}{b^2}\right)-a^2(\eta_{21}+\eta_{22}\eta_{21})\left(1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}\right)\right\},\\
s_z &= -\tfrac{1}{4}\omega^2\rho(\eta_{21}a^2+\eta_{22}b^2)/E_3,\\
\sigma_{xy} &= 2Nxy/n_3,\\
\sigma_{xz} &= \sigma_{yz} = 0
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(95);$$

where $N$ is given by (84).

§ 30. The conclusion that the above solution applies to an elliptic cylinder may be justified as follows:

The terms containing $z$ in equations (87) contribute nothing to the body-stress equations because $d\widehat{zz}/dz$, &c., are of the order of small quantities here neglected; thus the expressions (94) for the stresses will satisfy the body-stress equations. (This is easily verified of course directly.)

Again over the cylindrical surface

$$x^2/a^2+y^2/b^2=1,$$

we have from (94)

$$\widehat{xx}=-2Na^2y^2/b^2,\quad \widehat{yy}=-2Nb^2x^2/a^2,\quad \widehat{xy}=2Nxy\ \ldots\ldots\ldots\ldots\ldots\ldots(96);$$

whence

$$(x/a^2)\,\widehat{xx}+(y/b^2)\,\widehat{xy}=0,$$
$$(x/a^2)\,\widehat{xy}+(y/b^2)\,\widehat{yy}=0.$$

The equations over the *curved* surface are thus completely satisfied.

Over the terminal planes $z=\pm l$ the normal stress $\widehat{zz}$ does not vanish everywhere, as it strictly ought to do, but instead we have

$$\iint\widehat{zz}\,dxdy=0.$$

Thus, according to the theory of equipollent systems of loading, the solution is satisfactory, except at points in the immediate vicinity of the terminal sections.

§ 31. The increments in the semi-axes $a$ and $b$ are given by the same formulae, viz. (92), as apply in the case of the elongated ellipsoid. The reduction in the half length $l$ of the cylinder is given by

$$\delta l/l=-\tfrac{1}{4}\omega^2\rho(\eta_{21}a^2+\eta_{22}b^2)/E_3\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(97).$$

Comparing (97) with (90), supposing $l=c$, we see that the shortening in a long cylinder is greater than the corresponding axial shortening in an elongated ellipsoid

in the ratio 3 : 2. This arises from the reduction in the strain $s_z$ near the ends of the axis of rotation in the case of the long ellipsoid.

If undashed letters refer as above to a long cylinder, and dashed letters refer to the *axial* thickness of a thin disk of the same material and elliptic section rotating at the same speed, we find comparing (97) with (70) of Sect. IV.

$$(\delta l/l) \div (\delta l'/l') = 1 - \frac{1}{2}\left\{\frac{a^4}{E_1} + \frac{b^4}{E_2} + a^2b^2\left(\frac{1}{n_3} + \frac{2\eta_{12}}{E_1}\right)\right\} \div \left(\frac{2a^4}{E_1} + \frac{2b^4}{E_2} + \frac{a^2b^2}{n_3}\right) \text{.....} (98).$$

Thus the reduction per unit of length in the axis of rotation is invariably less in the long cylinder than in the thin disk.

As in the case of the disk, the tangential stress $\widehat{u}$ in the plane of the cross-section has a very simple form at the surface. For if $p$ be the perpendicular from the central axis on the tangent plane at a point $x$, $y$ on the surface, we easily find from (96)

$$\widehat{u} = -2Na^2b^2/p^2 \text{.....} (99),$$

$N$ being given by (84). At least as a rule $N$ is negative and $\widehat{u}$ a traction. The formula (99) differs from the corresponding result for a thin disk only in the value of $N$ (cf. (71)).

We can easily attach a simple physical significance to $N$. Thus let $\widehat{u}_1$ and $\widehat{u}_2$ represent the minimum and maximum surface values of $\widehat{u}$, occurring respectively at the ends of the major and minor axes of the elliptic section, then

$$N = -\tfrac{1}{2}(\widehat{u}_1 + \widehat{u}_2) \div (a^2 + b^2) \text{.....} (100).$$

## SECTION VIII.

### LONG ROTATING CIRCULAR CYLINDER OF MATERIAL SYMMETRICAL ROUND THE AXIS.

§ 32. When the cylinder is solid, the solution can be obtained by putting

$$b = a, \quad E_2 = E_1 = E', \quad \&c.,$$

in the results of last section. When the cylinder is hollow, an independent investigation is necessary.

In obtaining the following results I made use of the solution* published in 1892 for the case of isotropy, recognising that the type would remain unchanged. As the method adopted is practically identical with that applied in Section V. to the circular disk, I pass at once to the results. The origin has been taken at the mid-point of

* *Camb. Phil. Soc. Proc.* Vol. VII. pp. 283—305.

the cylinder's axis, and $r$, $\phi$, $z$ are ordinary cylindrical coordinates. The expressions for the stresses are as follows:

$$\left.\begin{aligned}
\widehat{rr} &= \omega^2\rho\,\{3+\eta'-2\eta^2(E'/E)\}(a^2-r^2)(1-a'^2/r^2)\div\{8(1-\eta^2E'/E)\},\\
\widehat{\phi\phi} &= \omega^2\rho\,[\{3+\eta'-2\eta^2(E'/E)\}(a^2+a'^2+a^2a'^2/r^2)-(1+3\eta'+2\eta^2E'/E)\,r^2]\div\{8(1-\eta^2E'/E)\},\\
\widehat{zz} &= \omega^2\rho\,\eta\,(1+\eta')(a^2+a'^2-2r^2)\div\{4(1-\eta^2E'/E)\},\\
\widehat{r\phi} &= \widehat{rz} = \widehat{\phi z} = 0
\end{aligned}\right\}\qquad\text{.....(101).}$$

The displacements $w$ parallel to the $z$-axis, and $u$ along $r$, are given by

$$\left.\begin{aligned}
w &= -\,\omega^2\rho\,\eta\,(a^2+a'^2)\,z/(2E),\\
u &= \frac{1}{8}\frac{\omega^2\rho}{1-\eta^2E'/E}\left[\left\{\frac{(1-\eta')(3+\eta')}{E'}-\frac{4\eta^2}{E}\right\}(a^2+a'^2)\,r-(1+\eta')\left(\frac{1-\eta'}{E'}-\frac{2\eta^2}{E}\right)r^3\right.\\
&\qquad\left.+\frac{1+\eta'}{E'}(3+\eta'-2\eta^2E'/E)\,a^2a'^2/r\right]
\end{aligned}\right\}\qquad\text{.....(102).}$$

An alternative form for $u$, worth recording, is

$$u=\frac{1}{8}\frac{\omega^2\rho}{E'(1-\eta^2E'/E)}[(1-\eta')(3+\eta')(a^2+a'^2)\,r-(1-\eta'^2)\,r^3+(1+\eta')(3+\eta')\,a^2a'^2/r$$
$$-\,2\eta^2(E'/E)\{2(a^2+a'^2)\,r-(1+\eta')\,r^3+(1+\eta')\,a^2a'^2/r\}]\text{.....(103).}$$

I shall assume $1-\eta^2E'/E$ to be positive; if it could be zero the expressions for the stresses and displacements could become infinite.

§ 33. The solution, except when $\eta=0$, is dependent on the theory of equipollent systems of loading, in so far as we have to substitute for the exact surface equation

$$\widehat{zz}=0 \text{ over } z=\pm l,$$

$$\int_{a'}^{a}2\pi r\widehat{zz}\,dr=0.$$

If we put $a'=0$ in (101), (102) and (103) we obtain the correct values of the stresses and displacements in a solid cylinder of radius $a$. The stresses and strains, however, near the inner surface of a *nearly* solid cylinder are, as in the case of the disk, totally different from those at the same axial distance in a wholly solid cylinder of the same external radius.

Comparing (101), (102) and (103) with (77) and (78), we see that when $\eta$ or $E'/E$ vanishes the formulae for the stresses and displacements in the long cylinder and thin disk become identical. This is true irrespective of the absolute values of $\eta'$ or $E'$.

§ 34. The stress system (101) possesses several features of interest. The radial stress $\widehat{rr}$ is everywhere positive, or a traction, except at the surfaces, where it vanishes; it has its maximum value where

$$r=\sqrt{aa'}.$$

The orthogonal stress $\widehat{\phi\phi}$ is everywhere a traction. Its largest and smallest values occur

respectively at the inner and outer surfaces. Distinguishing these values by the suffixes $_i$ and $_0$, we have

$$\left.\begin{aligned}\widehat{\phi\phi}_i &= \tfrac{1}{2}\omega^2\rho\,(a^2+a'^2)+\tfrac{1}{4}\omega^2\rho\,(1+\eta')\,(a^2-a'^2)/(1-\eta^2E'/E),\\ \widehat{\phi\phi}_0 &= \tfrac{1}{2}\omega^2\rho\,(a^2+a'^2)-\tfrac{1}{4}\omega^2\rho\,(1+\eta')\,(a^2-a'^2)/(1-\eta^2E'/E)\end{aligned}\right\}\text{.....(104).}$$

This shows very clearly how $\widehat{\phi\phi}_i$ and $\widehat{\phi\phi}_0$ approach equality as the thickness of the cylinder wall diminishes.

The third principal stress $\widehat{zz}$, parallel to the axis of rotation, is a traction inside a pressure outside the cylindrical surface

$$r=\sqrt{\tfrac{1}{2}(a^2+a'^2)}.$$

The surface values of $\widehat{zz}$, using suffixes as above, are given by

$$\widehat{zz}_i=-\widehat{zz}_0=\tfrac{1}{4}\omega^2\rho\eta\,(1+\eta')\,(a^2-a'^2)/(1-\eta^2E'/E)\text{.....(105).}$$

The numerical equality of $\widehat{zz}_i$ and $\widehat{zz}_0$ seems curious.

The following relation is also a neat one

$$\widehat{zz}_i-\widehat{zz}_0=\eta\,(\widehat{\phi\phi}_i-\widehat{\phi\phi}_0)\text{ .....(106).}$$

It somewhat reminds one of the results (81) established for the annular disk.

§ 35. Coming to the displacements, we see from (102) that the cross-sections—unlike those of the disk—remain plane. Further, if $\delta l$ and $\delta l'$ denote the changes in the length of a hollow and a solid circular cylinder of equal length, the material, section, and velocity being the same, we have

$$\delta l/\delta l'=1+a'^2/a^2\text{ .....(107).}$$

The influence of rotation on the length thus increases notably as the wall of the hollow cylinder becomes thinner. Comparing the first of equations (102) with the last of equations (81) we see that the change per unit length in the length of a long hollow cylinder is the exact arithmetic mean of the changes per unit thickness in the rim thicknesses of a thin disk of the same section and material rotating with equal velocity.

For the increments in the radii of the two surfaces of the long cylinder we find from (102)

$$\left.\begin{aligned}\delta a/a &= (\omega^2\rho/4E')\,\{(1-\eta')\,a^2+(3+\eta')\,a'^2\},\\ \delta a'/a' &= (\omega^2\rho/4E')\,\{(3+\eta')\,a^2+(1-\eta')\,a'^2\}\end{aligned}\right\}\text{ .....(108),}$$

formulae in exact agreement with the corresponding results (79) for the annular disk.

A variety of interesting relationships exist amongst the different displacements. Thus if $A$ represent the cross-section $\pi\,(a^2-a'^2)$, and $t$ the wall thickness $a-a'$, we have from (102)

$$\left.\begin{aligned}(\delta a/a)+(\delta a'/a') &= \omega^2\rho\,(a^2+a'^2)/E',\\ -(dl/l)\div\{(\delta a/a)+(\delta a'/a')\} &= -\eta E'/(2E)=-\tfrac{1}{2}\eta'',\\ (\delta a'/a')-(\delta a/a) &= \omega^2\rho\,(a^2-a'^2)/(4n_2)\end{aligned}\right\}\text{.....(109),}$$

$$(\delta A/A)=\omega^2\rho\,(a^2+a'^2)\,(1-\eta')/(2E')\text{ .....(110),}$$

$$(\delta t/t)=\omega^2\rho\,\{(a-a')^2-\eta'\,(a+a')^2\}/(4E')\text{ .....(111).}$$

If the increments in $a$, $a'$ and $l$ could be measured, the relations (109) would give $E'$, $\eta''$ and $n_2$ immediately.

From (110) we see that the area of the cross-section of the material is always increased by rotation; while (111) shows us that the cylinder wall becomes thicker or thinner according as

$$\eta' < \text{ or } > (a-a')^2/(a+a')^2 \quad \text{.....(112).}$$

For $\eta' = 1/4$ the thickness of the wall is unchanged when

$$a' = a/3 \quad \text{.....(113),}$$

and it is reduced when $a'/a$ exceeds $1/3$.

§ 36. The case when $t/a$ is very small, or the cylinder wall very thin, merits separate attention. Of the stresses, $\widehat{rr}$ is then small compared to $\widehat{zz}$, while $\widehat{zz}$ in its turn is small compared to $\widehat{\phi\phi}$; and to a first approximation the sole stress left is

$$\widehat{\phi\phi} = \omega^2\rho a^2 \quad \text{.....(114).}$$

Under like conditions first approximations to the displacements are

$$\left.\begin{aligned} w &= -\omega^2\rho\eta a^2 z/E, \\ u &= \omega^2\rho a^3/E' \end{aligned}\right\} \quad \text{.....(115).}$$

# VIII. *On the Change of a System of Independent Variables.*

By E. G. GALLOP, M.A., Fellow of Gonville and Caius College.

[*Received* and *read* 20 February 1899.]

THE main object of this paper is to establish a symbolical form for the result of changing a system of $n$ independent variables in a partial differential coefficient. This form is a generalization of that obtained by Mr Leudesdorf* in the case of a single independent variable. The method adopted is the same as in a previous paper† in which I obtained another proof of Mr Leudesdorf's result. It consists in developing the formula given by Jacobi‡ for the reversion of series. An advantage of adopting this method is that the relation between the symbolical form and the fully developed form given by Sylvester§ and proved by Cayley‖ is readily perceived, so that from the symbolical form the developed form can be written down.

The first of the fundamental formulae of this paper is equation (10) of § 2. This formula may be developed in two ways. By the detailed work of §§ 1, 3 Sylvester's expanded form, just referred to, is deduced from it; in the succeeding sections it is developed into the symbolical formulae (21) and (22) of § 11, and (26), (27), (28), (29), (30) of §§ 14, 15. Of these formulae (29), or its equivalent (30), seems to be the most important. The crucial point in the establishment of these formulae is the proof of equations (18 *a*, *b*, *c*) in § 9. The somewhat complicated work of §§ 1, 3 is introduced for the sake of showing the connexion between the symbolical formulae and Sylvester's expanded form; it is not required in the subsequent developments. In § 17 a symbolical formula is obtained for the differentiation of implicit functions, and in § 18 this formula is applied to determine a solution of the general equation of infinite degree involving a dependent variable $y$ and an independent variable $x$, when the equation has been deprived of the constant term.

The remainder of the paper is taken up with applications of the symbolical formulae (26)...(30). In several papers¶ published in the *Proceedings of the London Mathematical*

* "Second Paper on Change of the Independent Variable," *Proc. Lond. Math. Soc.* Vol. XVIII.

† "Change of the Independent Variable in a Differential Coefficient," *Camb. Phil. Trans.* Vol. XVI.

‡ "De resolutione aequationum per series infinitas," *Crelle's Jour.* Vol. VI.; *Gesammelte Werke*, VI. pp. 26—61.

§ *Proc. Roy. Soc.* Vol. VII., 1855; and *Quar. Jour. Math.* Vol. I. 1857, with corrections.

‖ "Deuxième note sur une formule pour la réversion des séries," *Crelle*, Vol. LIV.; *Collected Works*, Vol. IV. 234.

¶ "On the Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants," Vol. XVIII. pp. 142—164.

"On Pure Ternary Reciprocants and Functions allied to them," Vol. XIX. pp. 6—23.

"On Cyclicants, or Pure Ternary Reciprocants, and allied Functions," Vol. XIX. pp. 377—405.

"On Projective Cyclic Concomitants, or Surface Differential Invariants," Vol. XX. pp. 131—160.

"On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables," Vol. XXII. pp. 79—104.

"On the Transformation of Linear Partial Differential Operators by Extended Linear Continuous Groups," Vol. XXIX. p. 466.

*Society*, Prof. Elliott has considered the theory of Cyclicants and Reciprocants, mostly for the case of two independent variables. It is shown in this paper how most of Prof. Elliott's general theorems can be obtained for any number of independent variables as deductions from the general symbolical formulae. In § 19 the result of the change of variables by the general linear transformation is exhibited in formula (39), which may be regarded as fundamental in the theory of Pure Cyclicants. Prof. Elliott has also considered the conditions which must be satisfied by the functions which are the generalized forms of ordinary Reciprocants when there are two dependent and two independent variables. In §§ 29—31 these conditions are deduced from the general formulae of the paper for any number of variables, and it is also shown that Prof. Elliott's conditions are not independent and may be reduced in number.

For the sake of conciseness all the general results of the paper are worked out for the case of three independent variables, but care has been taken to use only such methods as would be applicable to any number of independent variables.

*Summary of Contents.*


§ 1. Definition of the function $D_x^p D_y^q D_z^r (tu^j v^g w^h)$, and its development as required in § 3.

§ 2. Formula for the change of a system of independent variables obtained from Jacobi's Theorem; equation (10).

§ 3. Digression to deduce Sylvester's form from equation (10).

§ 4. Modified form of equation (10).

§ 5. Definition of the operators $\{U_1 U_{\xi_1}\}$, $\{U_1 U_{\eta_1}\}$,... $\{U, x\}$, $\{U, y\}$,...

§§ 6, 7. Commutative character of the operators $\{U, x\}$, $\{U, y\}$,...

§ 8. Effect of the operators $\{U_1 U_{\xi_1}\}$, $\{U_1 U_{\eta_1}\}$,... on the function $D_x^p D_y^q D_z^r (tu^j v^g w^h)$.

§ 9. First transformation of the result in equation (10) to a symbolical form; equations (19 *a*, *b*, *c*).

§ 10. The operators $\omega_{xy}$, $\omega_{xz}$,...; their alternants.

§ 11. Final symbolical formula for the change of the independent variables in a partial differential coefficient by means of an operator involving $\{U, x\}$, $\{U, y\}$,... $\omega_{xy}$, $\omega_{xz}$,...; equations (21, 22).

§ 12. Various forms for that part of the operator which involves the $\omega$'s; determination of the four forms for the case of two independent variables.

§ 13. Case of one independent variable.

§ 14. Definition of the operators $[U, x]$, $[U, y]$,..., $[U_1]$, $[U_2]$,...

§ 15. Modified form of the general symbolical formula in § 11; equations (27) to (30).
The operators $\Omega_{xy}$, $\Omega_{xz}$,...; their alternants.

§ 16. Symbolical formulae for the interchange of the dependent and independent variables.

§ 17. Symbolical formula for the differentiation of implicit functions.

§ 18. Solution, in symbolical form, of the general algebraical equation of infinite degree; or of a system of such equations.

§ 19. Application to the general linear transformation.

§ 20. Various forms of the result of linear transformation in the case of two independent variables; cyclical interchange of the dependent and independent variables.

§ 21. Alternants of the operators $\omega$ and $V_1$, $V_2$, $V_3$ in the case of the linear transformation.

§§ 22, 23. Application to the theory of Cyclicants.

§ 24. Proof of a theorem due to Prof. Elliott.

§§ 25, 26, 27. Effect of the linear transformation on the operators $V$ and $\omega$; cyclical interchange as a particular case.

§ 28. Example illustrating the use of the last results.

§ 29. Definition of $n$-ary reciprocants; conditions satisfied by such functions.

§ 30. Example of this class of reciprocants.

§ 31. Proof that the conditions of § 29 are not independent, and that they may be reduced in number.

§ 1. Let $u$, $v$, $w$, $t$ be functions of three independent variables $x$, $y$, $z$; and let any differential coefficient

$$\frac{\partial^{p+q+r} u}{\partial x^p \partial y^q \partial z^r}$$

be denoted by $u_{pqr}$; let also the quantities

$$\frac{u_{pqr}}{p!\,q!\,r!}, \quad \frac{v_{pqr}}{p!\,q!\,r!}, \quad \frac{w_{pqr}}{p!\,q!\,r!}, \quad \frac{t_{pqr}}{p!\,q!\,r!}$$

be denoted by $a_{pqr}$, $b_{pqr}$, $c_{pqr}$, $d_{pqr}$.

Further, let $D_x^p D_y^q D_z^r (tu^f v^g w^h)$ denote the result of suppressing all terms which explicitly contain $t$, $u$, $v$, $w$ or first differential coefficients of $u$, $v$, $w$ in

$$\frac{\partial^{p+q+r}}{\partial x^p \partial y^q \partial z^r} (tu^f v^g w^h).$$

The expanded form of this expression is easily obtained. Let $\xi$, $\eta$, $\zeta$ be quantities independent of $x$, $y$, $z$; let

$$U = a_{200}\xi^2 + a_{020}\eta^2 + a_{002}\zeta^2 + a_{110}\xi\eta + \ldots + a_{200}\xi^2 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

let $V$ and $W$ denote similar expressions with $b$'s and $c$'s instead of $a$'s, and let

$$T = d_{100}\xi + d_{010}\eta + d_{001}\zeta + d_{200}\xi^2 + \ldots + d_{200}\xi^2 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Then

$$TU^f V^g W^h = \Sigma \frac{\xi^p \eta^q \zeta^r}{p!\,q!\,r!} D_\xi^p D_\eta^q D_\zeta^r (TU^f V^g W^h)$$

where, after the differentiations indicated by $D_\xi$, $D_\eta$, $D_\zeta$, $\xi$, $\eta$, $\zeta$ are made to vanish.

But obviously, on this understanding,

$$D_\xi^p D_\eta^q D_\zeta^r (TU^f V^g W^h) = D_x^p D_y^q D_z^r (tu^f v^g w^h).$$

Therefore

$$\frac{D_x^p D_y^q D_z^r (tu^f v^g w^h)}{p!\,q!\,r!}$$

is equal to the coefficient of $\xi^p \eta^q \zeta^r$ in $TU^f V^g W^h$. Therefore by ordinary algebra, provided $p+q+r > 2f+2g+2h$,

$$\begin{aligned}\frac{D_x^p D_y^q D_z^r (tu^f v^g w^h)}{p!\,q!\,r!\,f!\,g!\,h!} = \Sigma d_{\lambda\mu\nu} &\frac{1}{f_1!\,f_2!\ldots} (a_{\lambda_1\mu_1\nu_1})^{f_1} (a_{\lambda_2\mu_2\nu_2})^{f_2} \ldots \\ &\frac{1}{g_1!\,g_2!\ldots} (b_{\lambda_1'\mu_1'\nu_1'})^{g_1} (b_{\lambda_2'\mu_2'\nu_2'})^{g_2} \ldots \\ &\frac{1}{h_1!\,h_2!\ldots} (c_{\lambda_1''\mu_1''\nu_1''})^{h_1} (c_{\lambda_2''\mu_2''\nu_2''})^{h_2} \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),\end{aligned}$$

where summation extends to all zero and positive integral values of $f_1, f_2, \ldots g_1, g_2, \ldots h_1, h_2, \ldots \lambda, \mu, \nu, \lambda_1, \mu_1, \nu_1, \ldots$ which satisfy the conditions

$$\left.\begin{array}{ll} f_1+f_2+\ldots=f, \quad g_1+g_2+\ldots=g, \quad h_1+h_2+\ldots=h & \\ \lambda+\Sigma f_1\lambda_1+\Sigma g_1\lambda_1'+\Sigma h_1\lambda_1''=p, & \lambda+\mu+\nu \nless 1, \\ \mu+\Sigma f_1\mu_1+\Sigma g_1\mu_1'+\Sigma h_1\mu_1''=q, & \lambda_1+\mu_1+\nu_1 \nless 2, \\ \nu+\Sigma f_1\nu_1+\Sigma g_1\nu_1'+\Sigma h_1\nu_1''=r, & \lambda_1'+\mu_1'+\nu_1' \nless 2, \text{ \&c.} \end{array}\right\} \ldots\ldots(4).$$

Therefore, writing $[p, q, r; f, g, h; \lambda, \mu, \nu; \lambda_1, \mu_1, \nu_1;$ &c.] for the typical term on the right hand of (3), we have

$$\frac{D_x^p D_y^q D_z^r (tu^f v^g w^h)}{p!\,q!\,r!\,f!\,g!\,h!} = \Sigma\,[p, q, r; f, g, h; \lambda, \mu, \nu; \lambda_1, \mu_1, \nu_1; \text{ \&c.}] \ldots\ldots\ldots (5).$$

Again, on differentiating the equation

$$\frac{U^f}{f!} = \Sigma \frac{1}{f_1!\,f_2!\ldots}(a_{\lambda_1\mu_1\nu_1})^{f_1}(a_{\lambda_2\mu_2\nu_2})^{f_2}\ldots \xi^{\Sigma f_1\lambda_1}\eta^{\Sigma f_1\mu_1}\zeta^{\Sigma f_1\nu_1}$$

with respect to $\xi$, we find

$$\frac{U^{f-1}U_\xi}{(f-1)!} = \Sigma\,.\,\Sigma f_1\lambda_1\,.\,\frac{1}{f_1!\,f_2!\ldots}(a_{\lambda_1\mu_1\nu_1})^{f_1}\ldots \xi^{\Sigma f_1\lambda_1-1}\eta^{\Sigma f_1\mu_1}\zeta^{\Sigma f_1\nu_1}.$$

But

$$TU^{f-1}V^gW^hU_\xi = \Sigma D_x^p D_y^q D_z^r (tu^{f-1}v^gw^hu_x)\frac{\xi^p\eta^q\zeta^r}{p!\,q!\,r!};$$

therefore, by comparison of coefficients of powers of $\xi, \eta, \zeta$,

$$\frac{D_x^{p-1}D_y^qD_z^r(tu^{f-1}v^gw^hu_x)}{(p-1)!\,q!\,r!\,(f-1)!\,g!\,h!} = \Sigma\,.\,\Sigma f_1\lambda_1\,.\,[p, q, r; f, g, h; \lambda, \mu, \nu; \lambda_1, \mu_1, \nu_1; \text{ \&c.}]\ldots(6).$$

And similarly

$$\frac{D_x^{p-1}D_y^{q-1}D_z^r(tu^{f-1}v^{g-1}w^hu_xv_y)}{(p-1)!\,(q-1)!\,r!\,(f-1)!\,(g-1)!\,h!} = \Sigma\,.\,\Sigma f_1\lambda_1\,.\,\Sigma g_1\mu_1'\,[p, q, r; f, g, h; \lambda, \mu, \nu; \lambda_1, \mu_1, \nu_1; \text{ \&c.}]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

$$\frac{D_x^{p-1}D_y^{q-1}D_z^{r-1}(tu^{f-1}v^{g-1}w^{h-1}u_xv_yw_z)}{(p-1)!\,(q-1)!\,(r-1)!\,(f-1)!\,(g-1)!\,(h-1)!} = \Sigma\,.\,\Sigma f_1\lambda_1\,.\,\Sigma g_1\mu_1'\,.\,\Sigma h_1\nu_1''$$
$$[p, q, r; f, g, h; \lambda, \mu, \nu; \lambda_1, \mu_1, \nu_1; \text{ \&c.}]\ldots\ldots\ldots\ldots\ldots(8),$$

where the limits of summation are the same as in (4); and similar expressions can be written down with $x, y, z$ interchanged.

It is obvious that, if $p+q+r \leqq 2(f+g+h)$,

$$D_x^pD_y^qD_z^r(tu^fv^gw^h) = 0,$$

since the coefficient of $\xi^p\eta^q\zeta^r$ in $TU^fV^gW^h$ will be zero. For similar reasons

$$D_x^pD_y^qD_z^r(tu^fv^gw^hu_x) = 0, \text{ if } p+q+r \leqq 2f+2g+2h+1,$$
$$D_x^pD_y^qD_z^r(tu^fv^gw^hu_xv_y) = 0, \text{ if } p+q+r \leqq 2f+2g+2h+2, \text{ \&c.}$$

§ 2. We now proceed to establish the formula for the change of the independent variables in a partial differential coefficient. It is first necessary to state the theorem of Jacobi on which the method is based. Let $\upsilon$, $\nu$, $\omega$ be three quantities given in terms of three independent variables $\xi$, $\eta$, $\zeta$ by equations of the form

$$\upsilon = \xi + a_{200}\xi^2 + a_{020}\eta^2 + a_{002}\zeta^2 + a_{110}\xi\eta + \ldots + a_{200}\xi^2 + \ldots$$
$$= \xi + X, \text{ say};$$
$$\nu = \eta + b_{200}\xi^2 + \ldots + b_{200}\xi^2 + \ldots = \eta + Y,$$
$$\omega = \zeta + c_{200}\xi^2 + \ldots + c_{200}\xi^2 + \ldots = \zeta + Z,$$

where the $a$'s, $b$'s and $c$'s are any quantities independent of $\xi$, $\eta$, $\zeta$. It is important to remark that the linear part of each of the expressions must consist of a single term. If these equations are solved for $\xi$, $\eta$, $\zeta$ in terms of $\upsilon$, $\nu$, $\omega$ one set of values will vanish when $\upsilon$, $\nu$, $\omega$ vanish, and can be expanded in series proceeding by integral powers of $\upsilon$, $\nu$, $\omega$. Supposing that $\xi$, $\eta$, $\zeta$ have these values, we can then expand a general function $f(\xi, \eta, \zeta)$ in powers of $\upsilon$, $\nu$, $\omega$, and Jacobi's theorem states that the coefficient of $\upsilon^l\nu^m\omega^n$ in the result is equal to the coefficient of $\xi^{-1}\eta^{-1}\zeta^{-1}$ in the expansion of

$$f(\xi, \eta, \zeta)\frac{\partial(\upsilon, \nu, \omega)}{\partial(\xi, \eta, \zeta)}\frac{1}{(\xi+X)^{l+1}(\eta+Y)^{m+1}(\zeta+Z)^{n+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where the expansion is effected by first arranging $(\xi+X)^{-(l+1)}$, &c. in powers of $X/\xi$, $Y/\eta$, $Z/\zeta$ and then substituting for $X$, $Y$, $Z$ and multiplying together the various terms.

Now let $u$, $v$, $w$, $t$ be given functions of three independent variables $x$, $y$, $z$; and let it be required to change the variables from $x$, $y$, $z$ to $u$, $v$, $w$ and express $\partial^{l+m+n}t/\partial u^l\partial v^m\partial w^n$ in terms of differential coefficients of $t$, $u$, $v$, $w$ with respect to $x$, $y$, $z$. To this end let $x$, $y$, $z$ receive increments $\xi$, $\eta$, $\zeta$, and let the consequent increments in $u$, $v$, $w$, $t$ be $\upsilon$, $\nu$, $\omega$, $\tau$. The first differential coefficients of $u$, $v$, $w$, $t$ will be denoted by special letters according to the following scheme

$$u_{100} = a, \qquad u_{010} = a', \qquad u_{001} = a'',$$
$$v_{100} = b, \qquad v_{010} = b', \qquad v_{001} = b'',$$
$$w_{100} = c, \qquad w_{010} = c', \qquad w_{001} = c'',$$
$$t_{100} = d, \qquad t_{010} = d', \qquad t_{001} = d''.$$

Then

$$\upsilon = a\xi + a'\eta + a''\zeta + U,$$
$$\nu = b\xi + b'\eta + b''\zeta + V,$$
$$\omega = c\xi + c'\eta + c''\zeta + W,$$
$$\tau = T,$$

where $U$, $V$, $W$, $T$ have the same values as in the previous section.

But if $t$ be regarded as a function of $u$, $v$, $w$, we have also

$$\tau = \Sigma \frac{\partial^{l+m+n} t}{\partial u^l \partial v^m \partial w^n} \frac{\upsilon^l \nu^m \omega^n}{l!\ m!\ n!},$$

and therefore $\partial^{l+m+n}t/\partial u^l \partial v^m \partial w^n$ may be determined as the coefficient of $\upsilon^l \nu^m \omega^n / l!\ m!\ n!$ in the expansion of $\tau$ in powers of $\upsilon$, $\nu$, $\omega$.

To apply Jacobi's theorem we make the transformation

$$\xi' = a\xi + a'\eta + a''\zeta,$$
$$\eta' = b\xi + b'\eta + b''\zeta,$$
$$\zeta' = c\xi + c'\eta + c''\zeta;$$

so that

$$J\xi = A\xi' + B\eta' + C\zeta',$$
$$J\eta = A'\xi' + B'\eta' + C'\zeta',$$
$$J\zeta = A''\xi' + B''\eta' + C''\zeta',$$

where

$$J = \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix},$$

and $A$, $B$, $C$, ... are the first minors of $J$. We now have

$$\upsilon = \xi' + U', \quad \nu = \eta' + V', \quad \omega = \zeta' + W', \quad \tau = T',$$

where $U'$, $V'$, $W'$, $T'$ are the values of $U$, $V$, $W$, $T$ in terms of $\xi'$, $\eta'$, $\zeta'$.

To express these values take $D_\xi$, $D_\eta$, $D_\zeta$ as in § 1, and write

$$D_{\xi'} = \frac{1}{J}(AD_\xi + A'D_\eta + A''D_\zeta),$$

$$D_{\eta'} = \frac{1}{J}(BD_\xi + B'D_\eta + B''D_\zeta),$$

$$D_{\zeta'} = \frac{1}{J}(CD_\xi + C'D_\eta + C''D_\zeta),$$

and denote $D_{\xi'}{}^l D_{\eta'}{}^m D_{\zeta'}{}^n U'$, &c., by $u'_{lmn}$, &c.; and $u'_{lmn}/l!\ m!\ n!$ by $a'_{lmn}$, &c.

Then

$$U' = \Sigma a'_{pqr}\, \xi'^p \eta'^q \zeta'^r,$$
$$V' = \Sigma b'_{pqr}\, \xi'^p \eta'^q \zeta'^r,$$
$$W' = \Sigma c'_{pqr}\, \xi'^p \eta'^q \zeta'^r,$$
$$T' = \Sigma d'_{pqr}\, \xi'^p \eta'^q \zeta'^r,$$

where in $U'$, $V'$, $W'$, $p+q+r \nless 2$, and in $T'$, $p+q+r \nless 1$.

If also we take $D_x$, $D_y$, $D_z$ as in § 1 and write

$$D_1 = \frac{1}{J}(AD_x + A'D_y + A''D_z),$$

$$D_2 = \frac{1}{J}(BD_x + B'D_y + B''D_z),$$

$$D_3 = \frac{1}{J}(CD_x + C'D_y + C''D_z),$$

then

$$D_{\xi'}{}^p D_{\eta'}{}^q D_{\zeta'}{}^r (T'U'^f V'^g W'^h) = D_1^p D_2^q D_3^r (tu^f v^g w^h),$$

provided that products of the operators $D_1$, $D_2$, $D_3$ are formed by mere algebraical multiplication; that is to say, in the product $D_1 D_2$ it is supposed that $D_1$ does not operate on the coefficients of $D_x$, $D_y$, $D_z$ which occur in $D_2$.

Therefore by Jacobi's formula

$$\frac{1}{l!\,m!\,n!}\,\frac{\partial^{l+m+n} t}{\partial u^l \partial v^m \partial w^n}$$

is equal to the coefficient of $\xi'^{-1}\eta'^{-1}\zeta'^{-1}$ in

$$T'\frac{\partial(v,\ \nu,\ \omega)}{\partial(\xi',\ \eta',\ \zeta')}\,\frac{1}{(\xi'+U')^{l+1}(\eta'+V')^{m+1}(\zeta'+W')^{n+1}},$$

or

$$\Sigma_{f=0,\,g=0,\,h=0}^{f=\infty,\,g=\infty,\,h=\infty}(-1)^{f+g+h}\frac{(l+f)!\,(m+g)!\,(n+h)!}{l!\,f!\,m!\,g!\,n!\,h!}$$

$$\begin{vmatrix} 1+\dfrac{\partial U'}{\partial \xi'}, & \dfrac{\partial V'}{\partial \xi'}, & \dfrac{\partial W'}{\partial \xi'} \\ \dfrac{\partial U'}{\partial \eta'}, & 1+\dfrac{\partial V'}{\partial \eta'}, & \dfrac{\partial W'}{\partial \eta'} \\ \dfrac{\partial U'}{\partial \zeta'}, & \dfrac{\partial V'}{\partial \zeta'}, & 1+\dfrac{\partial W'}{\partial \zeta'} \end{vmatrix} \frac{T'U'^f V'^g W'^h}{\xi'^{l+f+1}\eta'^{m+g+1}\zeta'^{n+h+1}};$$

that is, in

$$\Sigma_{f=0,\,g=0,\,h=0}^{f=\infty,\,g=\infty,\,h=\infty}(-1)^{f+g+h}\frac{(l+f)!\,(m+g)!\,(n+h)!}{l!\,f!\,m!\,g!\,n!\,h!}$$

$$.\ \Sigma D_1^p D_2^q D_3^r \begin{vmatrix} 1+u_1, & v_1, & w_1 \\ u_2, & 1+v_2, & w_2 \\ u_3, & v_3, & 1+w_3 \end{vmatrix} tu^f v^g w^h\,.\,\frac{\xi'^{p-l-f-1}\eta'^{q-m-g-1}\zeta'^{r-n-h-1}}{p!\,q!\,r!},$$

where $u_1$, $v_1$, $w_1$, &c. stand for $D_1 u$, $D_1 v$, $D_1 w$, &c.

To obtain the term containing $\xi'^{-1}\eta'^{-1}\zeta'^{-1}$ we take $p = l+f$, $q = m+g$, $r = n+h$.

Hence

$$\frac{\partial^{l+m+n} t}{\partial u^l \partial v^m \partial w^n} = \Sigma_{f=0,\, g=0,\, h=0}^{f=\infty,\, g=\infty,\, h=\infty} (-1)^{f+g+h} D_1^{l+f} D_2^{m+g} D_3^{n+h}$$

$$\cdot \begin{vmatrix} 1+u_1, & v_1, & w_1 \\ u_2, & 1+v_2, & w_2 \\ u_3, & v_3, & 1+w_3 \end{vmatrix} t \frac{u^f v^g w^h}{f!\, g!\, h!} \quad \text{..... (10).}$$

Although in this expression summation extends to all values of $f$, $g$, $h$, it is obvious from the remark at the end of § 1 that terms for which $f+g+h \geqq l+m+n$ will be zero. In particular, if $l=1$, $m=0$, $n=0$, the only term not zero is that for which $f=0$, $g=0$, $h=0$.

It is easily verified that

$$\begin{vmatrix} 1+u_1, & v_1, & w_1 \\ u_2, & 1+v_2, & w_2 \\ u_3, & v_3, & 1+w_3 \end{vmatrix} = \begin{vmatrix} A/J, & B/J, & C/J \\ A'/J, & B'/J, & C'/J \\ A''/J, & B''/J, & C''/J \end{vmatrix} \times \begin{vmatrix} a+u_x, & b+v_x, & c+w_x \\ a'+u_y, & b'+v_y, & c'+w_y \\ a''+u_z, & b''+v_z, & c''+w_z \end{vmatrix}$$

$$= \frac{1}{J} \begin{vmatrix} a+u_x, & b+v_x, & c+w_x \\ a'+u_y, & b'+v_y, & c'+w_y \\ a''+u_z, & b''+v_z, & c''+w_z \end{vmatrix} \quad \text{..... (11),}$$

$$= \begin{vmatrix} u_x, & u_y, & u_z, & -1, & 0, & 0 \\ v_x, & v_y, & v_z, & 0, & -1, & 0 \\ w_x, & w_y, & w_z, & 0, & 0, & -1 \\ 1, & 0, & 0, & \frac{A}{J}, & \frac{B}{J}, & \frac{C}{J} \\ 0, & 1, & 0, & \frac{A'}{J}, & \frac{B'}{J}, & \frac{C'}{J} \\ 0, & 0, & 1, & \frac{A''}{J}, & \frac{B''}{J}, & \frac{C''}{J} \end{vmatrix} \quad \text{..... (12).}$$

It must be understood that when the determinant (11) or (12) is substituted in the expression (10) the operators $D_1$, $D_2$, $D_3$ affect $u_x$, $v_x$, $w_x$, &c. but not $a$, $b$, $c$, &c., or $A$, $B$, $C$, &c.

§ 3. To obtain Sylvester's expanded form of the result we use the form (12) for the determinant and expand $D_1^{l+f} D_2^{m+g} D_3^{n+h}$ in powers of $D_x$, $D_y$, $D_z$. This operator is equal to

$$\Sigma \frac{(l+f)!}{p_1!\, p_2!\, p_3!} \frac{(m+g)!}{q_1!\, q_2!\, q_3!} \frac{(n+h)!}{r_1!\, r_2!\, r_3!}$$

$$\left(\frac{AD_x}{J}\right)^{p_1} \left(\frac{A'D_y}{J}\right)^{p_2} \left(\frac{A''D_z}{J}\right)^{p_3} \left(\frac{BD_x}{J}\right)^{q_1} \left(\frac{B'D_y}{J}\right)^{q_2} \left(\frac{B''D_z}{J}\right)^{q_3} \left(\frac{CD_x}{J}\right)^{r_1} \left(\frac{C'D_y}{J}\right)^{r_2} \left(\frac{C''D_z}{J}\right)^{r_3},$$

where summation extends to all zero or positive integral values of $p_1$, $p_2$, $p_3$, &c. for which

$$p_1+p_2+p_3=l+f,\quad q_1+q_2+q_3=m+g,\quad r_1+r_2+r_3=n+h.$$

We now re-arrange the grouping of the terms and transform (10) into

$$\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n}=\Sigma_{f=0,\,g=0,\,h=0}^{f=\infty,\,g=\infty,\,h=\infty}(-1)^{f+g+h-3}D_1^{l+f-1}D_2^{m+g-1}D_3^{n+h-1}$$

$$\cdot\begin{vmatrix}(uD_x), & (uD_y), & (uD_z), & D_1, & 0, & 0\\ (vD_x), & (vD_y), & (vD_z), & 0, & D_2, & 0\\ (wD_x), & (wD_y), & (wD_z), & 0, & 0, & D_3\\ 1, & 0, & 0, & \frac{A}{J}, & \frac{B}{J}, & \frac{C}{J}\\ 0, & 1, & 0, & \frac{A'}{J}, & \frac{B'}{J}, & \frac{C'}{J}\\ 0, & 0, & 1, & \frac{A''}{J}, & \frac{B''}{J}, & \frac{C''}{J}\end{vmatrix}t\frac{u^fv^gw^h}{f!\,g!\,h!},$$

where $(uD_x)$, $(uD_y)$, $(uD_z)$ are the same in effect as $D_x$, $D_y$, $D_z$ but operate on $u$ only, whilst $(vD_x),\ldots$ operate on $v$ only, and $(wD_x),\ldots$ on $w$ only.

We can now make use of the results of § 1 and obtain finally

$$\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n}=\Sigma_{f=0,\,g=0,\,h=0}^{f=\infty,\,g=\infty,\,h=\infty}(-1)^{f+g+h-3}\frac{(l+f-1)!}{p_1!\,p_2!\,p_3!}\frac{(m+g-1)!}{q_1!\,q_2!\,q_3!}\frac{(n+h-1)!}{r_1!\,r_2!\,r_3!}$$

$$\times\Sigma(p_1+q_1+r_1-1)!\,(p_2+q_2+r_2-1)!\,(p_3+q_3+r_3-1)!$$

$$\times J^{-(l+m+n+f+g+h)}A^{p_1}A'^{p_2}A''^{p_3}B^{q_1}B'^{q_2}B''^{q_3}C^{r_1}C'^{r_2}C''^{r_3}$$

$$\times[(p_1+q_1+r_1),\ (p_2+q_2+r_2),\ (p_3+q_3+r_3);\ f,\ g,\ h;\ \lambda,\ \mu,\ \nu;\ \lambda_1,\ \mu_1,\ \nu_1;\ \&c.]$$

$$\times\begin{vmatrix}\Sigma f_1\lambda_1, & \Sigma f_1\mu_1, & \Sigma f_1\nu_1, & l+f, & 0, & 0\\ \Sigma g_1\lambda_1', & \Sigma g_1\mu_1', & \Sigma g_1\nu_1', & 0, & m+g, & 0\\ \Sigma h_1\lambda_1'', & \Sigma h_1\mu_1'', & \Sigma h_1\nu_1'', & 0, & 0, & n+h\\ p_1+q_1+r_1, & 0, & 0, & p_1, & q_1, & r_1\\ 0, & p_2+q_2+r_2, & 0, & p_2, & q_2, & r_2\\ 0, & 0, & p_3+q_3+r_3, & p_3, & q_3, & r_3\end{vmatrix}\quad\ldots\ldots\ldots\ldots(13),$$

where the limits of summation are given by (4).

A little consideration is needed to see the truth of the last result. It is obtained by regarding the determinant as expanded, then expanding the various terms by § 1 and grouping together all the terms which give rise to the particular term denoted by

$$[(p_1+q_1+r_1),\ (p_2+q_2+r_2),\ (p_3+q_3+r_3);\ f,\ g,\ h;\ \lambda,\ \mu,\ \nu;\ \lambda_1,\ \mu_1,\ \nu_1;\ \&c.].$$

The last result agrees with that given by Sylvester but differs in sign from that obtained by Cayley when the number of independent variables is odd.

In the case of certain terms, such as those which occur in the evaluation of $\frac{\partial^2 t}{\partial u^2}$, the coefficients as given by (13) take an indeterminate form, but as Sylvester explains, there is no difficulty in deducing the proper value.

§ 4. In order to obtain a symbolical expression for the result of the change of variables we return to equation (10), and by a re-arrangement of terms write it in the form

$$\frac{\partial^{l+m+n} t}{\partial u^l \partial v^m \partial w^n} = \Sigma_{f=0,\, g=0,\, h=0}^{f=\infty,\, g=\infty,\, h=\infty} (-1)^{f+g+h} D_1^{l+f-1} D_2^{m+g-1} D_3^{n+h-1}$$

$$\begin{vmatrix} D_1-(uD_1), & -(vD_1), & -(wD_1) \\ -(uD_2), & D_2-(vD_2), & -(wD_2) \\ -(uD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix} t \frac{u^f v^g w^h}{f!\,g!\,h!} \quad \text{.....} (14),$$

where, as in the previous section, $(uD_1)$, $(uD_2)$, $(uD_3)$ are equivalent to $D_1$, $D_2$, $D_3$ but operate on $u$ only. This re-arrangement is effected by grouping together terms homogeneous in $D_1$, $D_2$, $D_3$, $(uD_1)$, $(uD_2)$, .....

The operators required for the purpose of expressing the result symbolically will be considered in the following sections.

§ 5. Taking $U$, $V$, $W$, $T$ as in § 1, let suffixes 1, 2, 3, 4 indicate that $(\xi_1, \eta_1, \zeta_1)$, $(\xi_2, \eta_2, \zeta_2)$, ... are substituted in them for $\xi$, $\eta$, $\zeta$; so that

$$U_1 = a_{200}\xi_1^2 + a_{020}\eta_1^2 + \dots$$

Let also $U_{\xi_1}$, $U_{\xi_2}$, ... denote $\frac{\partial}{\partial \xi_1} U_1$, $\frac{\partial}{\partial \xi_2} U_2$, .... Let $\{U_1 U_{\xi_1}\}$ denote an operator formed by replacing terms such as $\xi_1^p \eta_1^q \zeta_1^r$ in the product $U_1 U_{\xi_1}$ by $\frac{\partial}{\partial a_{pqr}}$. The particular brackets { } will be used to indicate this operator and to distinguish it from a mere algebraical product. Similarly let $\{U_2 V_{\xi_2}\}$ be an operator formed from $U_2 V_{\xi_2}$ by replacing $\xi_2^p \eta_2^q \zeta_2^r$ by $\frac{\partial}{\partial b_{pqr}}$. Let also $\{U_3 W_{\xi_3}\}$ and $\{U_4 T_{\xi_4}\}$ be formed by replacing $\xi_3^p \eta_3^q \zeta_3^r$ by $\frac{\partial}{\partial c_{pqr}}$ and $\xi_4^p \eta_4^q \zeta_4^r$ by $\frac{\partial}{\partial d_{pqr}}$.

We shall also suppose eight similar operators $\{V_1 U_{\xi_1}\}$, $\{V_2 V_{\xi_2}\}$, ... to be formed in like manner, and twenty-four others by writing $\eta$ and $\zeta$ for $\xi$.

Written at full length for a few terms the first operator is

$$\begin{aligned}\{U_1 U_{\xi_1}\} = {} & 2a^2_{200} \frac{\partial}{\partial a_{200}} + a_{020} a_{110} \frac{\partial}{\partial a_{020}} + a_{002} a_{101} \frac{\partial}{\partial a_{003}} + 3a_{200} a_{110} \frac{\partial}{\partial a_{210}} + (2a_{200} a_{020} + a^2_{110}) \frac{\partial}{\partial a_{120}} \\ & + 3a_{200} a_{101} \frac{\partial}{\partial a_{201}} + (2a_{200} a_{002} + a^2_{101}) \frac{\partial}{\partial a_{102}} + (a_{020} a_{101} + a_{011} a_{110}) \frac{\partial}{\partial a_{021}} \\ & + (a_{110} a_{002} + a_{011} a_{101}) \frac{\partial}{\partial a_{012}} + (2a_{200} a_{011} + 2a_{110} a_{101}) \frac{\partial}{\partial a_{111}} \\ & + \dots\end{aligned}$$

Similarly

$$\{U_2V_{\xi_2}\} = 2a_{200}b_{200}\frac{\partial}{\partial b_{200}} + a_{020}b_{110}\frac{\partial}{\partial b_{020}} + a_{002}b_{101}\frac{\partial}{\partial b_{002}} + (a_{200}b_{110} + 2a_{110}b_{200})\frac{\partial}{\partial b_{110}} + \dots$$

If it is desired to work with $u_{pqr}$, $v_{pqr}$, &c. instead of $a_{pqr}$, $b_{pqr}$, &c., the operators may be formed in similar fashion. Thus $\{U_1U_{\xi_1}\}$ is formed from

$$\left(u_{200}\frac{\xi_1^2}{2!} + u_{020}\frac{\eta_1^2}{2!} + u_{002}\frac{\zeta_1^2}{2!} + u_{110}\xi_1\eta_1 + \dots + u_{300}\frac{\xi_1^3}{3!} + \dots\right)\left(u_{200}\xi_1 + u_{110}\eta_1 + u_{101}\zeta_1 + u_{300}\frac{\xi_1^2}{2!} + \dots\right)$$

by replacing $\xi_1^p\eta_1^q\zeta_1^r$ by $p!\,q!\,r!\,\dfrac{\partial}{\partial u_{pqr}}$.

And therefore the operators may be also expressed in the following manner

$$\{U_1U_{\xi_1}\} = \Sigma\, D_x^pD_y^qD_z^r(uu_x)\frac{\partial}{\partial u_{pqr}},$$

$$\{U_2V_{\xi_2}\} = \Sigma\, D_x^pD_y^qD_z^r(uv_x)\frac{\partial}{\partial v_{pqr}},$$

$$\{U_3W_{\xi_3}\} = \Sigma\, D_x^pD_y^qD_z^r(uw_x)\frac{\partial}{\partial w_{pqr}},$$

$$\{U_4T_{\xi_4}\} = \Sigma\, D_x^pD_y^qD_z^r(ut_x)\frac{\partial}{\partial t_{pqr}},$$

&c., &c.,

where summation may be supposed to extend to all positive integral values of $p$, $q$, $r$, though in the first three operators the coefficients of $\dfrac{\partial}{\partial u_{pqr}}$, $\dfrac{\partial}{\partial v_{pqr}}$, $\dfrac{\partial}{\partial w_{pqr}}$ will be zero if $p+q+r<3$, and in the fourth the coefficient of $\dfrac{\partial}{\partial t_{pqr}}$ will be zero if $p+q+r<2$.

The operators actually required will be nine formed by combinations of the above, viz.,

$$\{U, x\} = \{U_1U_{\xi_1}\} + \{U_2V_{\xi_2}\} + \{U_3W_{\xi_3}\} + \{U_4T_{\xi_4}\},$$

$$\{U, y\} = \{U_1U_{\eta_1}\} + \{U_2V_{\eta_2}\} + \{U_3W_{\eta_3}\} + \{U_4T_{\eta_4}\},$$

&c., &c.

§ 6. The first theorem to be established with regard to these operators $\{U, x\}$, $\{U, y\}, \dots$ is that they are all commutative with one another. But before proceeding to the proof of this theorem it is necessary to make a few preliminary remarks.

Let $F(\xi, \eta, \zeta)$ be any integral function consisting of terms $\xi^p\eta^q\zeta^r$ such that $p+q+r \nless 2$ and let $\{F(\xi_1, \eta_1, \zeta_1)\}$, $\{F(\xi_2, \eta_2, \zeta_2)\}, \dots$ be operators formed as in the preceding section.

It is then obvious that

$$\{F(\xi_1, \eta_1, \zeta_1)\}\, U_1 = F(\xi_1, \eta_1, \zeta_1),$$

$$\{F(\xi_1, \eta_1, \zeta_1)\}\, U_2 = F(\xi_2, \eta_2, \zeta_2),$$

$$\begin{aligned}\{F(\xi_1, \eta_1, \zeta_1)\}\, U_{\xi_1} &= \{F(\xi_1, \eta_1, \zeta_1)\} \frac{\partial}{\partial \xi_1} U_1 \\ &= \frac{\partial}{\partial \xi_1} \{F(\xi_1, \eta_1, \zeta_1)\}\, U_1 \\ &= \frac{\partial}{\partial \xi_1} F(\xi_1, \eta_1, \zeta_1),\end{aligned}$$

$$\{F(\xi_1, \eta_1, \zeta_1)\}\, U_{\xi_2} = \frac{\partial}{\partial \xi_2} F(\xi_2, \eta_2, \zeta_2),$$

$$\{F(\xi_1, \eta_1, \zeta_1)\}\, V_1 = 0.$$

A number of other similar results might be written down by interchanging suffixes and the quantities $U$, $V$, $W$, $T$, $\xi$, $\eta$, $\zeta$. Again

$$\begin{aligned}\{F(\xi_1, \eta_1, \zeta_1)\}\, U_1 U_{\xi_1} &= U_{\xi_1} \{F(\xi_1, \eta_1, \zeta_1)\}\, U_1 + U_1 \{F(\xi_1, \eta_1, \zeta_1)\}\, U_{\xi_1} \\ &= U_{\xi_1} F(\xi_1, \eta_1, \zeta_1) + U_1 \frac{\partial}{\partial \xi_1} F(\xi_1, \eta_1, \zeta_1);\end{aligned}$$

$$\begin{aligned}\{F(\xi_2, \eta_2, \zeta_2)\}\, U_1 V_{\xi_1} &= U_1 \{F(\xi_2, \eta_2, \zeta_2)\}\, V_{\xi_1} \\ &= U_1 \frac{\partial}{\partial \xi_1} F(\xi_1, \eta_1, \zeta_1).\end{aligned}$$

Many similar results could be obtained, but these will be sufficient to indicate the mode of procedure about to be adopted for forming the products of the operators

$$\{U, x\},\ \{U, y\}, \ldots\ldots.$$

§ 7. The essentially distinct cases to be considered are the products $\{U, x\}\{U, y\}$, $\{U, x\}\{V, x\}$, $\{U, x\}\{V, y\}$. We will take these cases in order.

Let $$\{U, x\}\{U, y\} = \{U, x\} \cdot \{U, y\} + \{U, x\} * \{U, y\},$$

where the first term on the right is the result of algebraical multiplication, and the second is the result of operating with $\{U, x\}$ on the coefficients of $\{U, y\}$. It is only the second term that can possibly be unsymmetrical. We have

$$\begin{aligned}\{U, x\} * \{U, y\} &= [\{U_1 U_{\xi_1}\} + \{U_2 V_{\xi_2}\} + \{U_3 W_{\xi_3}\} + \{U_4 T_{\xi_4}\}] * [\{U_1 U_{\eta_1}\} + \{U_2 V_{\eta_2}\} + \{U_3 W_{\eta_3}\} + \{U_4 T_{\eta_4}\}] \\ &= \{U_1 U_{\xi_1}\} * \{U_1 U_{\eta_1}\} + \{U_1 U_{\xi_1}\} * \{U_2 V_{\eta_2}\} + \{U_1 U_{\xi_1}\} * \{U_3 W_{\eta_3}\} + \{U_1 U_{\xi_1}\} * \{U_4 T_{\eta_4}\} \\ &\quad + \{U_2 V_{\xi_2}\} * \{U_2 V_{\eta_2}\} + \{U_3 W_{\xi_3}\} * \{U_3 W_{\eta_3}\} + \{U_4 T_{\xi_4}\} * \{U_4 T_{\eta_4}\} \\ &= \{U_{\eta_1} U_1 U_{\xi_1}\} + \left\{U_1 \frac{\partial}{\partial \eta_1} (U_1 U_{\xi_1})\right\} + \{V_{\eta_2} U_2 U_{\xi_2}\} + \{W_{\eta_3} U_3 U_{\xi_3}\} + \{T_{\eta_4} U_4 U_{\xi_4}\} \\ &\quad + \left\{U_2 \frac{\partial}{\partial \eta_2} (U_2 V_{\xi_2})\right\} + \left\{U_3 \frac{\partial}{\partial \eta_3} (U_3 W_{\xi_3})\right\} + \left\{U_4 \frac{\partial}{\partial \eta_4} (U_4 T_{\xi_4})\right\}\end{aligned}$$

$$= \{U_1(U_1U_{\xi_1\eta_1} + 2U_{\xi_1}U_{\eta_1})\} + \{U_2(U_2V_{\xi_2\eta_2} + U_{\xi_2}V_{\eta_2} + U_{\eta_2}V_{\xi_2})\}$$
$$+ \{U_3(U_3W_{\xi_3\eta_3} + U_{\xi_3}W_{\eta_3} + U_{\eta_3}W_{\xi_3})\} + \{U_4(U_4T_{\xi_4\eta_4} + U_{\xi_4}T_{\eta_4} + U_{\eta_4}T_{\xi_4})\}$$
$$= \{U, y\} * \{U, x\},$$

from the symmetry of the expression with regard to $\xi$ and $\eta$.

Again we have

$$\{U, x\} * \{V, x\} = [\{U_1U_{\xi_1}\} + \{U_2V_{\xi_2}\} + \{U_3W_{\xi_3}\} + \{U_4T_{\xi_4}\}] * [\{V_1U_{\xi_1}\} + \{V_2V_{\xi_2}\} + \{V_3W_{\xi_3}\} + \{V_4T_{\xi_4}\}]$$
$$= \{U_1U_{\xi_1}\} * \{V_1U_{\xi_1}\} + \{U_2V_{\xi_2}\} * \{V_1U_{\xi_1}\} + \{U_2V_{\xi_2}\} * \{V_2V_{\xi_2}\} + \{U_2V_{\xi_2}\} * \{V_3W_{\xi_3}\}$$
$$+ \{U_2V_{\xi_2}\} * \{V_4T_{\xi_4}\} + \{U_3W_{\xi_3}\} * \{V_3W_{\xi_3}\} + \{U_4T_{\xi_4}\} * \{V_4T_{\xi_4}\}$$
$$= \left\{V_1\frac{\partial}{\partial\xi_1}(U_1U_{\xi_1})\right\} + \{U_{\xi_1}U_1V_{\xi_1}\} + \{V_{\xi_2}U_2V_{\xi_2}\} + \left\{V_2\frac{\partial}{\partial\xi_2}(U_2V_{\xi_2})\right\}$$
$$+ \{W_{\xi_3}U_3V_{\xi_3}\} + \{T_{\xi_4}U_4V_{\xi_4}\} + \left\{V_3\frac{\partial}{\partial\xi_3}(U_3W_{\xi_3})\right\} + \left\{V_4\frac{\partial}{\partial\xi_4}(U_4T_{\xi_4})\right\}$$
$$= \{V_1U_1U_{\xi_1\xi_1} + V_1U^2_{\xi_1} + U_1U_{\xi_1}V_{\xi_1}\} + \{V_2U_2V_{\xi_2\xi_2} + V_2U_{\xi_2}V_{\xi_2} + U_2V^2_{\xi_2}\}$$
$$+ \{U_3V_3W_{\xi_3\xi_3} + W_{\xi_3}(V_3U_{\xi_3} + U_3V_{\xi_3})\} + \{U_4V_4T_{\xi_4\xi_4} + T_{\xi_4}(U_4V_{\xi_4} + V_4U_{\xi_4})\}$$
$$= \{V, x\} * \{U, x\},$$

from the symmetry of the result with respect to $U$ and $V$.

Finally

$$\{U, x\} * \{V, y\} = [\{U_1U_{\xi_1}\} + \{U_2V_{\xi_2}\} + \{U_3W_{\xi_3}\} + \{U_4T_{\xi_4}\}] * [\{V_1U_{\eta_1}\} + \{V_2V_{\eta_2}\} + \{V_3W_{\eta_3}\} + \{V_4T_{\eta_4}\}]$$
$$= \{U_1U_{\xi_1}\} * \{V_1U_{\eta_1}\} + \{U_2V_{\xi_2}\} * \{V_1U_{\eta_1}\} + \{U_2V_{\xi_2}\} * \{V_2V_{\eta_2}\} + \{U_2V_{\xi_2}\} * \{V_3W_{\eta_3}\}$$
$$+ \{U_2V_{\xi_2}\} * \{V_4T_{\eta_4}\} + \{U_3W_{\xi_3}\} * \{V_3W_{\eta_3}\} + \{U_4T_{\xi_4}\} * \{V_4T_{\eta_4}\}$$
$$= \left\{V_1\frac{\partial}{\partial\eta_1}(U_1U_{\xi_1})\right\} + \{U_{\eta_1}U_1V_{\xi_1}\} + \{V_{\eta_2}U_2V_{\xi_2}\} + \left\{V_2\frac{\partial}{\partial\eta_2}(U_2V_{\xi_2})\right\} + \{W_{\eta_3}U_3V_{\xi_3}\}$$
$$+ \{T_{\eta_4}U_4V_{\xi_4}\} + \left\{V_3\frac{\partial}{\partial\eta_3}(U_3W_{\xi_3})\right\} + \left\{V_4\frac{\partial}{\partial\eta_4}(U_4T_{\xi_4})\right\}$$
$$= \{U_1V_1U_{\xi_1\eta_1} + V_1U_{\xi_1}U_{\eta_1} + U_1U_{\eta_1}V_{\xi_1}\} + \{U_2V_2V_{\xi_2\eta_2} + V_2U_{\eta_2}V_{\xi_2} + U_2V_{\eta_2}V_{\xi_2}\}$$
$$+ \{U_3V_3W_{\xi_3\eta_3} + W_{\xi_3}V_3U_{\eta_3} + W_{\eta_3}U_3V_{\xi_3}\} + \{U_4V_4T_{\xi_4\eta_4} + T_{\xi_4}V_4U_{\eta_4} + T_{\eta_4}U_4V_{\xi_4}\}.$$

This expression is unchanged when $U$, $V$ and $\xi$, $\eta$ are interchanged and likewise the suffixes 1, 2. Therefore

$$\{U, x\} * \{V, y\} = \{V, y\} * \{U, x\}.$$

By interchanges of $U$, $V$, $W$ and $x$, $y$, $z$, the products of all pairs of the nine operators can be reduced to one or other of the three preceding. It has therefore been fully established that all the nine operators $\{U, x\}$, $\{U, y\}$, $\{U, z\}$, $\{V, x\}$, $\{V, y\}$, $\{V, z\}$, $\{W, x\}$, $\{W, y\}$, $\{W, z\}$ are commutative with one another.

§ 8. Now, taking $U$, $V$, $W$, $T$ as in § 1 we have

$$\{U_1U_{\xi_1}\}\, U^f = fU^{f-1}\{U_1U_{\xi_1}\}\, U = fU^{f-1}\,.\, UU_\xi = U\frac{\partial}{\partial\xi}(U^f) \quad\text{.....(15 a)},$$

$$\{U_1U_{\xi_1}\}\frac{\partial}{\partial\xi}U^f = \frac{\partial}{\partial\xi}\{U_1U_{\xi_1}\}\, U^f = \frac{\partial}{\partial\xi}\left(U\frac{\partial}{\partial\xi}U^f\right) \quad\text{.....(15 b)},$$

$$\{U_1U_{\xi_1}\}\frac{\partial}{\partial\eta}U^f = \frac{\partial}{\partial\eta}\{U_1U_{\xi_1}\}\, U^f = \frac{\partial}{\partial\eta}\left(U\frac{\partial}{\partial\xi}U^f\right) \quad\text{.....(15 c)},$$

$$\{U_2V_{\xi_2}\}\, V^g = gV^{g-1}\{U_2V_{\xi_2}\}\, V = gV^{g-1}UV_\xi = U\frac{\partial}{\partial\xi}V^g \quad\text{.....(15 d)},$$

$$\{U_2V_{\xi_2}\}\frac{\partial}{\partial\xi}V^g = \frac{\partial}{\partial\xi}\{U_2V_{\xi_2}\}\, V^g = \frac{\partial}{\partial\xi}\left(U\frac{\partial}{\partial\xi}V^g\right) \quad\text{.....(15 e)},$$

$$\{U_2V_{\xi_2}\}\frac{\partial}{\partial\eta}V^g = \frac{\partial}{\partial\eta}\{U_2V_{\xi_2}\}\, V^g = \frac{\partial}{\partial\eta}\left(U\frac{\partial}{\partial\xi}V^g\right) \quad\text{.....(15 f)},$$

$$\{U_4T_{\xi_4}\}\, T = U\frac{\partial T}{\partial\xi} \quad\text{.....(15 g)}.$$

By comparison of the coefficients of $\xi^p\eta^q\zeta^r$ on the two sides of these equations the following formulae are deduced:

$$\{U_1U_{\xi_1}\}\, D_x^pD_y^qD_z^r u^f = D_x^pD_y^qD_z^r\left(u\frac{\partial}{\partial x}u^f\right) \quad\text{.....(16 a)},$$

$$\{U_1U_{\xi_1}\}\, D_x^pD_y^qD_z^r\frac{\partial}{\partial x}u^f = D_x^pD_y^qD_z^r\frac{\partial}{\partial x}\left(u\frac{\partial}{\partial x}u^f\right) \quad\text{.....(16 b)},$$

$$\{U_1U_{\xi_1}\}\, D_x^pD_y^qD_z^r\frac{\partial}{\partial y}u^f = D_x^pD_y^qD_z^r\frac{\partial}{\partial y}\left(u\frac{\partial}{\partial x}u^f\right) \quad\text{.....(16 c)},$$

$$\{U_2V_{\xi_2}\}\, D_x^pD_y^qD_z^r v^g = D_x^pD_y^qD_z^r\, u\frac{\partial}{\partial x}v^g \quad\text{.....(16 d)},$$

$$\{U_2V_{\xi_2}\}\, D_x^pD_y^qD_z^r\frac{\partial}{\partial x}v^g = D_x^pD_y^qD_z^r\frac{\partial}{\partial x}\left(u\frac{\partial}{\partial x}v^g\right) \quad\text{.....(16 e)},$$

$$\{U_2V_{\xi_2}\}\, D_x^pD_y^qD_z^r\frac{\partial}{\partial y}v^g = D_x^pD_y^qD_z^r\frac{\partial}{\partial y}\left(u\frac{\partial}{\partial x}v^g\right) \quad\text{.....(16 f)},$$

$$\{U_4T_{\xi_4}\}\, D_x^pD_y^qD_z^r t = D_x^pD_y^qD_z^r\left(u\frac{\partial t}{\partial x}\right) \quad\text{.....(16 g)}.$$

The form of the above results shows that they may be generalized by replacing $D_x^pD_y^qD_z^r$ by any function consisting of integral powers of $D_x$, $D_y$, $D_z$. These examples seem sufficient to show the effect of the operators. It will be noticed that the effect is to introduce a solitary $u$ and to make certain alterations in the symbols of differentiation. The effects are perhaps best seen by examination of (16 *e*) and (16 *f*).

With a view to the application of these formulae to the result in § 2 it is convenient to re-write them in another form. Let $(uD_x)$, as on previous occasions, represent $D_x$ when operating on $u$ only, and let $[uD_x]$ act only on the solitary $u$ which is introduced into the last set of formulae.

With this notation (16 $a$) becomes, if $F(D)$ represents any function of $D_x$, $D_y$, $D_z$,

$$\{U_1 U_{\xi_1}\} F(D) u^f = F(D) \{(uD_x) - [uD_x]\} u \,.\, u^f.$$

Moreover, since $\{U_1 U_{\xi_1}\}$ does not operate on $V$, $W$ or $T$, the equations (15 $a$, $b$, $c$) still hold if the functions operated on, viz. $U^f$, $\frac{\partial}{\partial \xi} U^f$, ..., are multiplied by powers of $V$, $W$, $T$ and their differential coefficients. Thus from (15 $b$) we have, for example,

$$\{U_1 U_{\xi_1}\} \,.\, \frac{\partial}{\partial \xi} U^f \,.\, V^g V_\xi W^h W_\eta T = \frac{\partial}{\partial \xi}\left(U \frac{\partial}{\partial \xi} U^f\right) . \, V^g V_\xi W^h W_\eta T;$$

and corresponding to (16 $a$)

$$\{U_1 U_{\xi_1}\} F(D) \phi(vD, wD, tD) u^f v^g w^h t = F(D) \phi(vD, wD, tD) \{(uD_x) - [uD_x]\} u \,.\, u^f v^g w^h t,$$

where $\phi(vD, wD, tD)$ represents a function of $(vD_x)$, $(vD_y)$, $(vD_z)$, $(wD_x)$, ....

Similarly

$$\{U_1 U_{\eta_1}\} F(D) \phi(vD, wD, tD) u^f v^g w^h t = F(D) \phi(vD, wD, tD) \{(uD_y) - [uD_y]\} u \,.\, u^f v^g w^h t,$$

$$\{U_1 U_{\zeta_1}\} F(D) \phi(vD, wD, tD) u^f v^g w^h t = F(D) \phi(vD, wD, tD) \{(uD_z) - [uD_z]\} u \,.\, u^f v^g w^h t,$$

and therefore

$$\left[\frac{A}{J}\{U_1 U_{\xi_1}\} + \frac{A'}{J}\{U_1 U_{\eta_1}\} + \frac{A''}{J}\{U_1 U_{\zeta_1}\}\right] F(D) \phi(vD, wD, tD) u^f v^g w^h t$$
$$= F(D) \phi(vD, wD, tD) \{(uD_1) - [uD_1]\} u \,.\, u^f v^g w^h t \text{.....(17 } a\text{)},$$

whilst similar results hold for $D_2$ and $D_3$.

Again, from formulae of which (16 $d$) is a type are deduced formulae exemplified by the following

$$\left[\frac{A}{J}\{U_2 V_{\xi_2}\} + \frac{A'}{J}\{U_2 V_{\eta_2}\} + \frac{A''}{J}\{U_2 V_{\zeta_2}\}\right] F(D) \phi(uD, wD, tD) u^f v^g w^h t$$
$$= F(D) \phi(uD, wD, tD) (vD_1) u \,.\, u^f v^g w^h t \text{.....(17 } b\text{)}.$$

And, from formulae of which (16 $e$) and (16 $f$) are types are deduced others which are exemplified by

$$\left[\frac{A}{J}\{U_2 V_{\xi_2}\} + \frac{A'}{J}\{U_2 V_{\eta_2}\} + \frac{A''}{J}\{U_2 V_{\zeta_2}\}\right] F(D) \phi(wD, tD) (vD_1) u^f v^g w^h t$$
$$= F(D) \phi(wD, tD) \{(vD_1) + [uD_1]\} (vD_1) u \,.\, u^f v^g w^h t \text{.....(17 } c\text{)}.$$

If in this last formula $(vD_1)$ is replaced on the left by $(vD_2)$, then on the right $\{(vD_1) + [uD_1]\}$ must be replaced by $\{(vD_2) + [uD_2]\}$; and if on the left $A$, $A'$, $A''$ are replaced by $B$, $B'$, $B''$, then on the right the second $(vD_1)$ must be replaced by $(vD_2)$.

§ 9. Return now to the expression (14) in § 4 and write, for brevity,

$$\begin{vmatrix} D_1 - (uD_1), & -(vD_1), & -(wD_1) \\ -(uD_2), & D_2 - (vD_2), & -(wD_2) \\ -(uD_3), & -(vD_3), & D_3 - (wD_3) \end{vmatrix} = |\Delta|.$$

By adding the second and third columns to the first it is evident that

$$|\Delta| = \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix}.$$

By use of this form and consideration of the rules exemplified in the preceding formulae it then becomes evident from (17 $a$) that

$$\left[\frac{A}{J}\{U_1U_{\xi_1}\}+\frac{A'}{J}\{U_1U_{\eta_1}\}+\frac{A''}{J}\{U_1U_{\zeta_1}\}\right] D_1^lD_2^mD_3^n\,|\Delta|\,u^fv^gw^ht$$

$$= D_1^lD_2^mD_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix} \{(uD_1)-[uD_1]\}\,u\,.\,u^fv^gw^ht.$$

Also from (17 $b$) and (17 $c$)

$$\left[\frac{A}{J}\{U_2V_{\xi_2}\}+\frac{A'}{J}\{U_2V_{\eta_2}\}+\frac{A''}{J}\{U_2V_{\zeta_2}\}\right] D_1^lD_2^mD_3^n\,|\Delta|\,u^fv^gw^ht$$

$$= D_1^lD_2^mD_3^n \begin{vmatrix} (tD_1), & -(vD_1)-[uD_1], & -(wD_1) \\ (tD_2), & D_2-(vD_2)-[uD_2], & -(wD_2) \\ (tD_3), & -(vD_3)-[uD_3], & D_3-(wD_3) \end{vmatrix} (vD_1)\,u\,.\,u^fv^gw^ht.$$

Again by interchange of $v$ and $w$ in (17 $b$) and (17 $c$) it is obvious that

$$\left[\frac{A}{J}\{U_3W_{\xi_3}\}+\frac{A'}{J}\{U_3W_{\eta_3}\}+\frac{A''}{J}\{U_3W_{\zeta_3}\}\right] D_1^lD_2^mD_3^n\,|\Delta|\,u^fv^gw^ht$$

$$= D_1^lD_2^mD_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1)-[uD_1] \\ (tD_2), & D_2-(vD_2), & -(wD_2)-[uD_2] \\ (tD_3), & -(vD_3), & D_3-(wD_3)-[uD_3] \end{vmatrix} (wD_1)\,u\,.\,u^fv^gw^ht.$$

Similarly by interchange of $v$ and $t$ in (17 $b$) and (17 $c$) it follows that

$$\left[\frac{A}{J}\{U_4T_{\xi_4}\}+\frac{A'}{J}\{U_4T_{\eta_4}\}+\frac{A''}{J}\{U_4T_{\zeta_4}\}\right] D_1^lD_2^mD_3^n\,|\Delta|\,u^fv^gw^ht$$

$$= D_1^lD_2^mD_3^n \begin{vmatrix} (tD_1)+[uD_1], & -(vD_1), & -(wD_1) \\ (tD_2)+[uD_2], & D_2-(vD_2), & -(wD_2) \\ (tD_3)+[uD_3], & -(vD_3), & D_3-(wD_3) \end{vmatrix} (tD_1)\,u\,.\,u^fv^gw^ht.$$

Now add these four equations together. The operator on the left will become

$$\frac{A}{J}\{U, x\}+\frac{A'}{J}\{U, y\}+\frac{A''}{J}\{U, z\},$$

which it will be convenient to denote by $\{U, 1\}$.

On the right-hand side all the terms containing $[uD_1]$, $[uD_2]$ and $[uD_3]$ disappear. For the coefficient of $[uD_1]$ in the operator is easily seen to be

$$-D_1^l D_2^m D_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix}$$

$$+D_1^l D_2^m D_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix}$$

$$=0.$$

The coefficient of $[uD_2]$ is

$$D_1^l D_2^m D_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_1), & -(vD_1), & -(wD_1) \\ (tD_3), & -(vD_3), & D_3-(wD_3) \end{vmatrix} = 0.$$

The coefficient of $[uD_3]$ is

$$D_1^l D_2^m D_3^n \begin{vmatrix} (tD_1), & -(vD_1), & -(wD_1) \\ (tD_2), & D_2-(vD_2), & -(wD_2) \\ (tD_1), & -(vD_1), & -(wD_1) \end{vmatrix} = 0.$$

Hence

$$\{U, 1\}\, D_1^l D_2^m D_3^n \,|\Delta|\, u^f v^g w^h t = D_1^l D_2^m D_3^n \,|\Delta|\, \{(uD_1)+(vD_1)+(wD_1)+(tD_1)\}\, u^{f+1} v^g w^h t$$

$$= D_1^{l+1} D_2^m D_3^n \,|\Delta|\, u^{f+1} v^g w^h t \quad \text{.....(18 } a\text{).}$$

We note that $\{U, 1\}$ may be written

$$\{U, 1\} = \frac{1}{J} \begin{vmatrix} \{U, x\}, & \{U, y\}, & \{U, z\} \\ b, & b', & b'' \\ c, & c', & c'' \end{vmatrix}.$$

If we take similarly

$$\{V, 2\} = \frac{1}{J} \begin{vmatrix} a, & a', & a'' \\ \{V, x\}, & \{V, y\}, & \{V, z\} \\ c, & c', & c'' \end{vmatrix} = \frac{1}{J}[B\{V, x\} + B'\{V, y\} + B''\{V, z\}],$$

and

$$\{W, 3\} = \frac{1}{J} \begin{vmatrix} a, & a', & a'' \\ b, & b', & b'' \\ \{W, x\}, & \{W, y\}, & \{W, z\} \end{vmatrix} = \frac{1}{J}[C\{W, x\} + C'\{W, y\} + C''\{W, z\}],$$

we shall find

$$\{V, 2\}\, D_1^l D_2^m D_3^n \,|\Delta|\, u^f v^g w^h t = D_1^l D_2^{m+1} D_3^n \,|\Delta|\, u^f v^{g+1} w^h t \quad \text{.....(18 } b\text{),}$$

$$\{W, 3\}\, D_1^l D_2^m D_3^n \,|\Delta|\, u^f v^g w^h t = D_1^l D_2^m D_3^{n+1} \,|\Delta|\, u^f v^g w^{h+1} t \quad \text{.....(18 } c\text{).}$$

We have therefore

$$D_1^{l+f-1}D_2^{m+g-1}D_3^{n+h-1}\,|\Delta|\,u^f v^g w^h t = \{U,\ 1\}\ D_1^{l+f-2}D_2^{m+g-1}D_3^{n+h-1}\,|\Delta|\,u^{f-1}v^g w^h t$$

$$= \{U,\ 1\}^f\ D_1^{l-1}D_2^{m+g-1}D_3^{n+h-1}\,|\Delta|\,v^g w^h t$$

$$= \{U,\ 1\}^f\,\{V,\ 2\}^g\,\{W,\ 3\}^h\ D_1^{l-1}D_2^{m-1}D_3^{n-1}\,|\Delta|\,t.$$

But $|\Delta|\,t = D_1D_2D_3t$, so that the expression becomes

$$\{U,\ 1\}^f\,\{V,\ 2\}^g\,\{W,\ 3\}^h\,.\,D_1^lD_2^mD_3^nt.$$

Therefore

$$\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n} = \sum_{f=0,\,g=0,\,h=0}^{f=\infty,\,g=\infty,\,h=\infty}(-1)^{f+g+h}D_1^{l+f-1}D_2^{m+g-1}D_3^{n+h-1}\,|\Delta|\,\frac{u^f v^g w^h t}{f!\,g!\,h!}$$

$$= \Sigma\,(-1)^{f+g+h}\,\frac{\{U,\ 1\}^f}{f!}\,\frac{\{V,\ 2\}^g}{g!}\,\frac{\{W,\ 3\}^h}{h!}\,.\,D_1^lD_2^mD_3^nt$$

$$= e^{-\{U,\,1\}-\{V,\,2\}-\{W,\,3\}}\,.\,D_1^lD_2^mD_3^nt \quad\text{.....(19 a)},$$

since the operators are commutative. The fact that the operators are commutative has been proved independently, but it is pretty obvious from the circumstance that the $D_1$, $D_2$, $D_3$ are commutative, so that the above reduction might have been effected in different orders.

If it is desired to bring the independent operators $\{U,\ x\}$, $\{U,\ y\}$, &c. into prominence the result may be written

$$\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n} = e^{-\frac{A}{J}\{U,\,x\}}\,.\,e^{-\frac{A'}{J}\{U,\,y\}}\,.\,e^{-\frac{A''}{J}\{U,\,z\}}\,.\,e^{-\frac{B}{J}\{V,\,x\}}\,.\,e^{-\frac{B'}{J}\{V,\,y\}}\,.\,e^{-\frac{B''}{J}\{V,\,z\}}$$

$$.\,e^{-\frac{C}{J}\{W,\,x\}}\,.\,e^{-\frac{C'}{J}\{W,\,y\}}\,.\,e^{-\frac{C''}{J}\{W,\,z\}}\,.\,D_1^lD_2^mD_3^n\,t \quad\text{.....(19 b)}.$$

If we write
$$D_{lmn} = \frac{1}{l!\ m!\ n!}\,\frac{\partial^{l+m+n}t}{\partial u^l\partial v^m\partial w^n},$$

we have
$$D_{lmn} = e^{-\{U,\,1\}-\{V,\,2\}-\{W,\,3\}}\,.\,\frac{D_1^lD_2^mD_3^n}{l!\ m!\ n!}\,t \quad\text{.....(19 c)}.$$

§ 10. It remains to express $\dfrac{D_1^lD_2^mD_3^nt}{l!\ m!\ n!}$ by means of operators acting upon $d_{lmn}$. Since $D_1^lD_2^mD_3^n$ is merely a linear transformation of $D_x^lD_y^mD_z^n$, the operators required for the purpose will be simply the ordinary operators of the theory of invariants. These operators we shall define as follows:—

$$\omega_{xy} = \{\xi T_\eta\} = \sum_{p=1,\,q=0,\,r=0}(q+1)\,d_{p-1,\,q+1,\,r}\frac{\partial}{\partial d_{pqr}},$$

$$\omega_{yx} = \{\eta T_\xi\} = \sum_{p=0,\,q=1,\,r=0}(p+1)\,d_{p+1,\,q-1,\,r}\frac{\partial}{\partial d_{pqr}},$$

$$\omega_{xz} = \{\xi T_\zeta\} = \sum_{p=1,\,q=0,\,r=0}(r+1)\,d_{p-1,\,q,\,r+1}\frac{\partial}{\partial d_{pqr}},$$

$$\omega_{zx} = \{\zeta T_\xi\} = \sum_{p=0,\, q=0,\, r=1} (p+1)\, d_{p+1,\, q,\, r-1} \frac{\partial}{\partial d_{pqr}},$$

$$\omega_{yz} = \{\eta T_\zeta\} = \sum_{p=0,\, q=1,\, r=0} (r+1)\, d_{p,\, q-1,\, r+1} \frac{\partial}{\partial d_{pqr}},$$

$$\omega_{zy} = \{\zeta T_\eta\} = \sum_{p=0,\, q=0,\, r=1} (q+1)\, d_{p,\, q+1,\, r-1} \frac{\partial}{\partial d_{pqr}},$$

where $T$ is as in (2), and the operators are formed by expanding $\xi T_\eta$, &c., and replacing $\xi^p \eta^q \zeta^r$ by $\dfrac{\partial}{\partial d_{pqr}}$. The upper limits of $p$, $q$, $r$ in the summations are all infinite.

These operators $\omega$ are identical with the operators $\Omega$ discussed in Elliott's *Algebra of Quantics*, Chap. XVI. Their properties are there obtained by forming the alternants, but as the formation is simplified by use of the symbolical method the process by this method is given here.

We remark that the operators $\partial/\partial\xi$ and $\{\xi T_\eta\}$ are independent and therefore, if $T_\xi$ denotes an algebraical expression,

$$\{\xi T_\eta\}\, T_\xi = \{\xi T_\eta\} \frac{\partial}{\partial \xi}\, T = \frac{\partial}{\partial \xi} \{\xi T_\eta\}\, T = \frac{\partial}{\partial \xi} (\xi T_\eta).$$

Hence

$$\begin{aligned}
\omega_{xy}\omega_{yz} - \omega_{yz}\omega_{xy} &= \{\xi T_\eta\}\{\eta T_\xi\} - \{\eta T_\xi\}\{\xi T_\eta\} \\
&= \left\{\eta \frac{\partial}{\partial \xi}(\xi T_\eta)\right\} - \left\{\xi \frac{\partial}{\partial \eta}(\eta T_\xi)\right\} \\
&= \{\eta T_\eta\} - \{\xi T_\xi\} \qquad \text{(20 a)},
\end{aligned}$$

$$\begin{aligned}
\omega_{xy}\omega_{zx} - \omega_{zx}\omega_{xy} &= \{\xi T_\eta\}\{\xi T_\zeta\} - \{\xi T_\zeta\}\{\xi T_\eta\} \\
&= \left\{\xi \frac{\partial}{\partial \zeta}(\xi T_\eta)\right\} - \left\{\xi \frac{\partial}{\partial \eta}(\xi T_\zeta)\right\} \\
&= 0 \qquad \text{(20 b)},
\end{aligned}$$

$$\begin{aligned}
\omega_{zy}\omega_{zx} - \omega_{zx}\omega_{zy} &= \{\xi T_\eta\}\{\zeta T_\xi\} - \{\zeta T_\xi\}\{\xi T_\eta\} \\
&= \left\{\zeta \frac{\partial}{\partial \xi}(\xi T_\eta)\right\} - \left\{\xi \frac{\partial}{\partial \eta}(\zeta T_\xi)\right\} \\
&= \{\zeta T_\eta\} = \omega_{zy} \qquad \text{(20 c)}.
\end{aligned}$$

Similarly

$$\omega_{xy}\omega_{yz} - \omega_{yz}\omega_{xy} = -\,\omega_{xz} \qquad \text{(20 d)},$$

$$\omega_{xy}\omega_{zy} - \omega_{zy}\omega_{xy} = 0 \qquad \text{(20 e)}.$$

Equation (20 $a$) shows that if a function is annihilated both by $\omega_{xy}$ and $\omega_{yz}$ and is isobaric in first suffixes it must also be isobaric in second suffixes and the partial weights must be equal, and if the function is further annihilated by $\omega_{xz}$ and $\omega_{zx}$ it must also be isobaric in third suffixes and all the partial weights must be equal. Equations (20 $b$) and (20 $e$) show that any two $\omega$'s are commutative if they have the

same letter for their first suffix, or the same letter for their second suffix. Equations (20 *c*) and (20 *d*) show that the $\omega$'s are not independent but that any one of them can be expressed as the alternant of two others, so that if a function is annihilated by the three operators $\omega_{yz}$, $\omega_{zx}$, $\omega_{xy}$, it is also annihilated by all the others.

When the case of more than three independent variables is considered there will be pairs of $\omega$'s which have no common letter in their suffixes, and reference to the above proof of equations (20) shows that such pairs are commutative.

That the operators $\{U, x\}$, $\{V, x\}$, ... are not commutative with any of the $\omega$'s is easily seen by forming the alternants of typical pairs. Thus

$$\begin{aligned}
\{U, x\}\,\omega_{xy} - \omega_{xy}\,\{U, x\} &= \{U_4 T_{\xi_4}\}\,\{\xi_4 T_{\eta_4}\} - \{\xi_4 T_{\eta_4}\}\,\{U_4 T_{\xi_4}\} \\
&= \left\{\xi_4 \frac{\partial}{\partial \eta_4}(U_4 T_{\xi_4})\right\} - \left\{U_4 \frac{\partial}{\partial \xi_4}(\xi_4 T_{\eta_4})\right\} \\
&= \{\xi_4 U_{\eta_4} T_{\xi_4}\} - \{U_4 T_{\eta_4}\}, \\
\{U, x\}\,\omega_{yz} - \omega_{yz}\,\{U, x\} &= \{U_4 T_{\xi_4}\}\,\{\eta_4 T_{\xi_4}\} - \{\eta_4 T_{\xi_4}\}\,\{U_4 T_{\xi_4}\} \\
&= \left\{\eta_4 \frac{\partial}{\partial \xi_4}(U_4 T_{\xi_4})\right\} - \left\{U_4 \frac{\partial}{\partial \xi_4}(\eta_4 T_{\xi_4})\right\} \\
&= \{\eta_4 U_{\xi_4} T_{\xi_4}\}, \\
\{U, x\}\,\omega_{yz} - \omega_{yz}\,\{U, x\} &= \{U_4 T_{\xi_4}\}\,\{\eta_4 T_{\zeta_4}\} - \{\eta_4 T_{\zeta_4}\}\,\{U_4 T_{\xi_4}\} \\
&= \left\{\eta_4 \frac{\partial}{\partial \zeta_4}(U_4 T_{\xi_4})\right\} - \left\{U_4 \frac{\partial}{\partial \xi_4}(\eta_4 T_{\zeta_4})\right\} \\
&= \{\eta_4 U_{\zeta_4} T_{\xi_4}\}.
\end{aligned}$$

§ 11. We can now express

$$\frac{D_1^l D_2^m D_3^n}{l!\,m!\,n!}\,t$$

by means of these operators acting upon $d_{lmn}$.

We have, in the first place,

$$\omega_{xz} d_{lmn} = (n+1)\, d_{l-1,\, m,\, n+1},$$

$$\omega_{xz}^p d_{lmn} = \frac{(n+p)!}{n!}\, d_{l-p,\, m,\, n+p},$$

$$\omega_{xy}^q \omega_{xz}^p d_{lmn} = \frac{(n+p)!\,(m+q)!}{n!\,m!}\, d_{l-p-q,\, m+q,\, n+p}.$$

Therefore

$$\begin{aligned}
\frac{D_1^l D_y^m D_z^n t}{l!\,m!\,n!} &= \frac{1}{l!\,m!\,n!}\left(\frac{A D_x + A' D_y + A'' D_z}{J}\right)^l D_y^m D_z^n t \\
&= \frac{1}{J^l}\,\frac{1}{l!\,m!\,n!}\,\Sigma\,\frac{l!}{p!\,q!\,l-p-q!}\, A^{l-p-q} A'^q A''^p\, D_x^{l-p-q}\, D_y^{m+q}\, D_z^{n+p}\, t
\end{aligned}$$

$$= \frac{1}{J^l}\frac{1}{m!\,n!}\Sigma\frac{1}{p!\,q!}A^{l-p-q}A'^q A''^p (m+q)!\,(n+p)!\,d_{l-p-q,\,m+q,\,n+p}$$

$$= \left(\frac{A}{J}\right)^l \Sigma\frac{1}{p!\,q!}\left(\frac{A'}{A}\right)^q\left(\frac{A''}{A}\right)^p \omega_{xy}{}^q\,\omega_{xz}{}^p\,d_{lmn}$$

$$= \left(\frac{A}{J}\right)^l e^{\frac{A'\omega_{xy}+A''\omega_{xz}}{A}}\,d_{lmn},$$

since $\omega_{xy}$ and $\omega_{xz}$ are commutative.

To express $D_1{}^l D_2{}^m D_z{}^n t$ in a similar manner we proceed thus. We have

$$JD_1 = AD_x + A'D_y + A''D_z,$$
$$JD_2 = BD_x + B'D_y + B''D_z;$$

whence
$$AD_2 = BD_1 + c''D_y - c'D_z.$$

It may be noticed here that, if we were dealing with more than three independent variables, $c''$ and $-c'$ would be replaced by second minors of $J$.

We have therefore

$$\frac{D_1{}^l D_2{}^m D_z{}^n t}{l!\,m!\,n!} = \frac{A^{-m}}{l!\,m!\,n!}D_1{}^l(BD_1 + c''D_y - c'D_z)^m D_z{}^n t$$

$$= \frac{A^{-m}}{l!\,m!\,n!}\Sigma\frac{m!}{p!\,q!\,r!}B^p D_1{}^{l+p}(c''D_y)^q(-c'D_z)^r D_z{}^n t, \text{ where } p+q+r=m,$$

$$= \frac{A^{-m}}{l!\,n!}\Sigma\frac{1}{p!\,q!\,r!}B^p c''^q(-c')^r(l+p)!\,q!\,(n+r)!\left(\frac{A}{J}\right)^{l+p} e^{\frac{A'\omega_{xy}+A''\omega_{xz}}{A}}\,d_{l+p,\,q,\,n+r}$$

$$= \left(\frac{A}{J}\right)^l\left(\frac{c''}{A}\right)^m e^{\frac{A'\omega_{xy}+A''\omega_{xz}}{A}}\Sigma\frac{1}{p!\,r!}\left(\frac{AB}{Jc''}\right)^p\left(-\frac{c'}{c''}\right)^r \omega_{yx}{}^p\,\omega_{yz}{}^r\,d_{lmn}$$

$$= \left(\frac{A}{J}\right)^l\left(\frac{c''}{A}\right)^m e^{\frac{A'\omega_{xy}+A''\omega_{xz}}{A}}\,.\,e^{\frac{AB}{Jc''}\omega_{yx}-\frac{c'}{c''}\omega_{yz}}\,.\,d_{lmn}.$$

Finally, since $D_z = a''D_1 + b''D_2 + c''D_3$, we have

$$\frac{D_1{}^l D_2{}^m D_3{}^n t}{l!\,m!\,n!} = \frac{D_1{}^l D_2{}^m}{l!\,m!\,n!}c''^{-n}(D_z - a''D_1 - b''D_2)^n t$$

$$= \frac{c''^{-n}}{l!\,m!\,n!}\Sigma\frac{n!}{p!\,q!\,r!}(-a'')^q(-b'')^r D_1{}^{l+q}D_2{}^{m+r}D_z{}^p t, \text{ where } p+q+r=n,$$

$$= \frac{c''^{-n}}{l!\,m!}\Sigma\frac{(-a'')^q(-b'')^r}{p!\,q!\,r!}\left(\frac{A}{J}\right)^{l+q}\left(\frac{c''}{A}\right)^{m+r} e^{\frac{A'\omega_{xy}+A''\omega_{xz}}{A}}\,e^{\frac{AB}{Jc''}\omega_{yx}-\frac{c'}{c''}\omega_{yz}}(l+q)!\,(m+r)!\,p!\,d_{l+q,\,m+r,\,p}$$

$$= \left(\frac{A}{J}\right)^l\left(\frac{c''}{A}\right)^m\frac{1}{c''^n}e^{\frac{A'\omega_{xy}+A''\omega_{xz}}{A}}\,.\,e^{\frac{AB}{Jc''}\omega_{yx}-\frac{c'}{c''}\omega_{yz}}\Sigma\frac{1}{q!\,r!}\left(-\frac{a''A}{J}\right)^q\left(-\frac{b''c''}{A}\right)^r(\omega_{zx})^q(\omega_{zy})^r\,.\,d_{lmn}$$

$$= \frac{A^{l-m}c''^{m-n}}{J^l}e^{\frac{A'\omega_{xy}+A''\omega_{xz}}{A}}\,.\,e^{\frac{AB}{Jc''}\omega_{yx}-\frac{c'}{c''}\omega_{yz}}\,.\,e^{-\frac{a''A}{J}\omega_{zx}-\frac{b''c''}{A}\omega_{zy}}\,.\,d_{lmn}.$$

Now, for brevity, write

$$\omega_1 = \frac{A'}{A}\,\omega_{xy} + \frac{A''}{A}\,\omega_{xz},$$

$$\omega_2 = \frac{AB}{Jc''}\,\omega_{yx} - \frac{c'}{c''}\,\omega_{yz},$$

$$\omega_3 = -\frac{a''A}{J}\,\omega_{zx} - \frac{b''c''}{A}\,\omega_{zy},$$

$$\begin{aligned}\{U, V, W\} &= \{U, 1\} + \{V, 2\} + \{W, 3\}\\ &= \frac{1}{J}[A\{U, x\} + A'\{U, y\} + A''\{U, z\}]\\ &\quad + \frac{1}{J}[B\{V, x\} + B'\{V, y\} + B''\{V, z\}]\\ &\quad + \frac{1}{J}[C\{W, x\} + C'\{W, y\} + C''\{W, z\}].\end{aligned}$$

Then

$$\begin{aligned}D_{lmn} &= \frac{1}{l!\,m!\,n!}\,\frac{\partial^{l+m+n}\,t}{\partial u^l \partial v^m \partial w^n}\\ &= \frac{A^{l-m}\,c''^{m-n}}{J^l}\,e^{-\{U, V, W\}}\,.\,e^{\omega_1}\,.\,e^{\omega_2}\,.\,e^{\omega_3}\,.\,d_{lmn} \quad\text{.....(21).}\end{aligned}$$

Now if $\Omega$ denotes any linear operator which acts on two functions $P$ and $Q$, we have

$$e^{\Omega}P\,.\,e^{\Omega}Q = e^{\Omega_1}\,.\,e^{\Omega_2}PQ,$$

where $\Omega_1$, $\Omega_2$ are equivalent to $\Omega$ but act respectively on $P$ and $Q$ alone. Therefore

$$\begin{aligned}e^{\Omega}P\,.\,e^{\Omega}Q &= e^{\Omega_1+\Omega_2}PQ\\ &= e^{\Omega}PQ.\end{aligned}$$

By repeated applications of this principle we find that

$$D_{lmn}D_{l'm'n'} = \frac{A^{l+l'-m-m'}\,c''^{m+m'-n-n'}}{J^{l+l'}}\,e^{-\{U, V, W\}}\,.\,e^{\omega_1}\,.\,e^{\omega_2}\,.\,e^{\omega_3}\,.\,d_{lmn}d_{l'm'n'}.$$

And more generally if $F(d_{lmn},\ d_{l'm'n'}, \ldots)$ represents any function isobaric in each set of suffixes being of weights $p_1$, $p_2$, $p_3$ in first, second and third suffixes,

$$F(D_{lmn},\ D_{l'm'n'}, \ldots) = \frac{A^{p_1-p_2}\,c''^{p_3-p_2}}{J^{p_1}}\,e^{-\{U, V, W\}}\,e^{\omega_1}e^{\omega_2}e^{\omega_3}\,F(d_{lmn},\ d_{l'm'n'}, \ldots) \quad\text{..... (22).}$$

§ 12. The asymmetry of that part of the operator which depends on the $\omega$'s is a consequence of their non-commutative character. By arranging the work a little differently nine different forms of the result could have been obtained. In the case of two independent variables the number of different forms will be four, and it will be convenient for some of the subsequent applications to have these four forms set out at length.

In modifying the work of § 11 for this case it is obvious by reference to the argument that only two $\omega$'s will be required, viz.

$$\omega_{xy} = \{\xi T_\eta\} = \sum_{p=1,\, q=0} (q+1)\, d_{p-1,\, q+1} \frac{\partial}{\partial d_{pq}},$$

$$\omega_{yx} = \{\eta T_\xi\} = \sum_{p=0,\, q=1} (p+1)\, d_{p+1,\, q-1} \frac{\partial}{\partial d_{pq}};$$

and their effect on $d_{lm}$ is seen to be this:—

$$\omega_{xy}{}^p d_{lm} = \frac{(m+p)!}{m!} d_{l-p,\, m+p},$$

$$\omega_{yx}{}^p d_{lm} = \frac{(l+p)!}{l!} d_{l+p,\, m-p}.$$

Moreover, reference to the work shows that $c''$ must be replaced by unity, and that

$$J = \begin{vmatrix} a & a' \\ b & b' \end{vmatrix}.$$

Therefore

$$\frac{D_1{}^l D_2{}^m t}{l!\, m!} = \left(\frac{A}{J}\right)^l \left(\frac{1}{A}\right)^m e^{\frac{A'}{A}\omega_{xy}} e^{\frac{AB}{J}\omega_{yx}} d_{lm} \quad \text{.....(23 a).}$$

We will next obtain the second form of the result. We have

$$D_1{}^l D_x{}^m t = \frac{1}{J^l}(AD_x + A'D_y)^l D_x{}^m t$$

$$= \frac{1}{J^l} \Sigma \frac{l!}{p!\, q!} A^p A'^q D_x{}^{m+p} D_y{}^q t, \text{ where } p+q=l.$$

Therefore

$$\frac{D_1{}^l D_x{}^m t}{l!\, m!} = \frac{1}{J^l} \Sigma \frac{(m+p)!}{m!\, p!} A^p A'^q d_{m+p,\, l-p}$$

$$= \left(\frac{A'}{J}\right)^l \Sigma \frac{1}{p!} \left(\frac{A}{A'}\right)^p \omega_{yx}{}^p d_{ml}$$

$$= \left(\frac{A'}{J}\right)^l e^{\frac{A}{A'}\omega_{yx}} d_{ml}.$$

Now

$$JD_1 = AD_x + A'D_y,$$

$$JD_2 = BD_x + B'D_y;$$

therefore, eliminating $D_y$,

$$A'D_2 = B'D_1 - D_x.$$

Hence

$$D_1^l D_2^m t = \frac{1}{A'^m}(B'D_1 - D_x)^m D_1^l t$$

$$= \frac{1}{A'^m}\Sigma \frac{m!}{p!\,q!} B'^p (-1)^q D_1^{l+p} D_x^q t$$

$$= \frac{1}{A'^m}\Sigma \frac{m!}{p!\,q!} B'^p (-1)^q (l+p)!\,q! \left(\frac{A'}{J}\right)^{l+p} e^{\frac{A}{A'}\omega_{yx}} d_{m-p,\, l+p}.$$

Therefore

$$\frac{D_1^l D_2^m t}{l!\,m!} = \left(\frac{A'}{J}\right)^l \left(-\frac{1}{A'}\right)^m e^{\frac{A}{A'}\omega_{yx}} \Sigma \frac{(l+p)!}{l!\,p!}\left(-\frac{A'B'}{J}\right)^p d_{m-p,\, l+p}$$

$$= \left(\frac{A'}{J}\right)^l \left(-\frac{1}{A'}\right)^m e^{\frac{A}{A'}\omega_{yx}} \Sigma \frac{1}{p!}\left(-\frac{A'B'}{J}\right)^p \omega_{xy}{}^p d_{ml}$$

$$= (-1)^m \frac{A'^{l-m}}{J^l} e^{\frac{A}{A'}\omega_{yx}} e^{-\frac{A'B'}{J}\omega_{xy}} d_{ml} \quad \text{..... (23 } b\text{).}$$

Interchanging $B$, $B'$ with $A$, $A'$ and $l$ with $m$, and writing $-J$ for $J$, we shall find similarly

$$\frac{D_1^l D_2^m t}{l!\,m!} = (-1)^l \left(\frac{B}{J}\right)^m \left(\frac{1}{B}\right)^l e^{\frac{B'}{B}\omega_{xy}} e^{-\frac{AB}{J}\omega_{yx}} d_{ml} \quad \text{..... (23 } c\text{)}$$

$$= \left(\frac{B'}{J}\right)^m \left(\frac{1}{B'}\right)^l e^{\frac{B}{B'}\omega_{yx}} e^{\frac{A'B'}{J}\omega_{xy}} d_{lm} \quad \text{..... (23 } d\text{).}$$

From these results it follows that, in the case of two independent variables, if $F$ is isobaric and of weights $p_1$, $p_2$ in first and second suffixes,

$$F(D_{lm}, \ldots) = K \,.\, e^{-\{UV\}} \,.\, e^{\omega_1} \,.\, e^{\omega_2} F(d_{lm}, \ldots) \quad \text{..... (24 } a\text{),}$$

where $$\{UV\} = \frac{1}{J}[A\{U, x\} + A'\{U, y\} + B\{V, x\} + B'\{V, y\}] \quad \text{..... (24 } b\text{),}$$

and the quantity $K$ and the operators $\omega_1$, $\omega_2$ may have either of the two sets of values

$$K = \frac{A^{p_1 - p_2}}{J^{p_1}}, \quad \omega_1 = \frac{A'}{A}\omega_{xy}, \quad \omega_2 = \frac{AB}{J}\omega_{yx} \quad \text{..... (24 } c\text{),}$$

$$K = \frac{B'^{p_2 - p_1}}{J^{p_2}}, \quad \omega_1 = \frac{B}{B'}\omega_{yx}, \quad \omega_2 = \frac{A'B'}{J}\omega_{xy} \quad \text{..... (24 } d\text{).}$$

And, as another form,

$$F(D_{lm}, \ldots) = Ke^{-\{UV\}} \,.\, e^{\omega_1} \,.\, e^{\omega_2} F(d_{ml}, \ldots) \quad \text{..... (25 } a\text{),}$$

where $K$, $\omega_1$, $\omega_2$ may have either of the two sets of values

$$K=(-1)^{p_2}\frac{A'^{p_1-p_2}}{J^{p_1}},\quad \omega_1=\frac{A}{A'}\omega_{yx},\quad \omega_2=-\frac{A'B'}{J}\omega_{xy}\ \ldots\ldots\ldots\ldots\ldots\ (25\ b),$$

$$K=(-1)^{p_1}\frac{B^{p_2-p_1}}{J^{p_2}},\quad \omega_1=\frac{B'}{B}\omega_{xy},\quad \omega_2=-\frac{AB}{J}\omega_{yx}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25\ c).$$

Here $A$, $A'$, $B$, $B'$ are the first minors of $J$, and therefore $A=b'$, $A'=-b$, $B=-a'$, $B'=a$.

§ 13. In the particular case when there is only one independent variable $x$ which is transformed to $u$, we have $J=u_x$, $A=1$,

$$\{U,\ V,\ W\}=\{U,\ x\}=\{U_1U_{\xi_1}\}+\{U_2T_{\xi_2}\};$$

there are no $\omega$'s, and we have, if $F$ is an isobaric function of weight $p$,

$$F(D_l,\ D_{l'},\ldots)=\frac{1}{u_x{}^p}e^{-\{U,\ x\}}F(d_l,\ d_{l'},\ \ldots),$$

and
$$D_l=\frac{1}{l\,!}\frac{\partial^l t}{\partial u^l}=\frac{1}{u_x{}^l}e^{-\{U,\ x\}}\,.\,d_l.$$

This form is not quite the same as that given by Mr Leudesdorf (*Proc. Lond. Math. Soc.*, Vol. XVIII.) and also established in my previous paper (*Trans. Camb. Phil. Soc.*, Vol. XVI.), but one formula can be deduced from the other by the method of the next section.

§ 14. Another form of the general result is often more useful than that stated in equation (21). It is obtained by exhibiting separately the terms containing first differential coefficients of $t$. For this purpose modified forms of the operators $\{U,\ x\}$, ... must be used; let $[U,\ x]$ denote the result of suppressing all terms in $\{U,\ x\}$ which contain $d_{100}$, $d_{010}$, $d_{001}$, so that $[U,\ x]$ may be formed in exactly the same way as $\{U,\ x\}$, except that in the process of formation the value of $T$ used is

$$d_{200}\xi^2+d_{020}\eta^2+d_{002}\zeta^2+d_{110}\xi\eta+\ldots$$

instead of that given in (2). Let $[V,\ x]$, $[W,\ x]$, $[U,\ y]$, ... represent similar modifications of $\{V,\ x\}$, $\{W,\ x\}$, $\{U,\ y\}$, .... Therefore

$$\{U,\ x\}=[U,\ x]+d_{100}[U_4],$$

$$\{U,\ y\}=[U,\ y]+d_{010}[U_4],$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where $[U_4]$ is formed by replacing $\xi_4{}^p\eta_4{}^q\zeta_4{}^r$ in $a_{200}\xi_4{}^2+\ldots$ with $\dfrac{\partial}{\partial d_{pqr}}$.

The twelve operators $[U_4]$, $[V_4]$, $[W_4]$, $[U, x]$, $[U, y]$, ... are easily seen to be all commutative with one another. For, by § 7,

$$\{U, x\}\{V, y\} - \{V, y\}\{U, x\} = 0;$$

therefore

$$\{[U, x] + d_{100}[U_4]\}\{[V, y] + d_{010}[V_4]\} - \{[V, y] + d_{010}[V_4]\}\{[U, x] + d_{100}[U_4]\} = 0.$$

Hence, by selection of the coefficients of $d_{100}$, $d_{010}$, $d_{100}d_{010}$, it follows that

$$\begin{aligned}
[U, x][V, y] - [V, y][U, x] &= 0,\\
[U_4][V, y] - [V, y][U_4] &= 0,\\
[U, x][V_4] - [V_4][U, x] &= 0,\\
[U_4][V_4] - [V_4][U_4] &= 0;
\end{aligned}$$

and in similar fashion it may be proved that the alternants of all other pairs of the operators are zero.

Now, by (19c),

$$D_{lmn} = e^{-\{U, V, W\}} \frac{D_1^l D_2^m D_3^n t}{l!\, m!\, n!}.$$

Therefore if $[U, V, W]$ is the modified form of $\{U, V, W\}$, so that

$$\begin{aligned}
[U, V, W] = &\frac{1}{J}(A[U, x] + A'[U, y] + A''[U, z])\\
&+ \frac{1}{J}(B[V, x] + B'[V, y] + B''[V, z])\\
&+ \frac{1}{J}(C[W, x] + C'[W, y] + C''[W, z]),
\end{aligned}$$

$$D_{lmn} = e^{-[U, V, W]} \cdot e^{-\frac{1}{J}(Ad + A'd' + A''d'')[U_4]} \cdot e^{-\frac{1}{J}(Bd + B'd' + B''d'')[V_4]} \cdot e^{-\frac{1}{J}(Cd + C'd' + C''d'')[W_4]} \frac{D_1^l D_2^m D_3^n t}{l!\, m!\, n!}.$$

Now $D_1^l D_2^m D_3^n t$ is a linear function of $d_{pqr}, \ldots$. Therefore the effect of $[U_4]$, $[V_4]$, $[W_4]$ operating on $D_1^l D_2^m D_3^n t$ is to change $d_{pqr}$ into $a_{pqr}$, $b_{pqr}$, $c_{pqr}$ and therefore to produce

$$D_1^l D_2^m D_3^n u, \quad D_1^l D_2^m D_3^n v, \quad D_1^l D_2^m D_3^n w;$$

whilst repeated operations by $[U_4]$, $[V_4]$, $[W_4]$ produce zero results.

Hence

$$\begin{aligned}
D_{lmn} = e^{-[U, V, W]} \frac{1}{J}[&J \,.\, D_1^l D_2^m D_3^n t - (Ad + A'd' + A''d'')\, D_1^l D_2^m D_3^n u\\
&- (Bd + B'd' + B''d'')\, D_1^l D_2^m D_3^n v - (Cd + C'd' + C''d'')\, D_1^l D_2^m D_3^n w]
\end{aligned}$$

$$= \frac{1}{J} e^{-[U, V, W]} \begin{vmatrix} D_1^l D_2^m D_3^n t, & D_1^l D_2^m D_3^n u, & D_1^l D_2^m D_3^n v, & D_1^l D_2^m D_3^n w \\ d, & a, & b, & c \\ d', & a', & b', & c' \\ d'', & a'', & b'', & c'' \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots (26).$$

§ 15. Up to the present there has been no restriction on $l$, $m$, $n$ except that they be not all zero; the last formula holds when $l+m+n=1$ on the understanding, assumed throughout, that $D_1^l D_2^m D_3^n u$, $D_1^l D_2^m D_3^n v$, $D_1^l D_2^m D_3^n w$ all vanish when $l+m+n=1$. But it is necessary to assume, in what follows, that $l+m+n>1$.

Corresponding to the operators $\omega$ of § 10 we introduce six operators $\Omega$ given by the equations

$$\begin{aligned}
\Omega_{xy} &= [\xi_1 U_{\eta_1}] + [\xi_2 V_{\eta_2}] + [\xi_3 W_{\eta_3}] + [\xi_4 T_{\eta_4}],\\
\Omega_{yx} &= [\eta_1 U_{\xi_1}] + [\eta_2 V_{\xi_2}] + [\eta_3 W_{\xi_3}] + [\eta_4 T_{\xi_4}],\\
\Omega_{xz} &= [\xi_1 U_{\zeta_1}] + [\xi_2 V_{\zeta_2}] + [\xi_3 W_{\zeta_3}] + [\xi_4 T_{\zeta_4}],\\
\Omega_{zx} &= [\zeta_1 U_{\xi_1}] + [\zeta_2 V_{\xi_2}] + [\zeta_3 W_{\xi_3}] + [\zeta_4 T_{\xi_4}],\\
\Omega_{yz} &= [\eta_1 U_{\zeta_1}] + [\eta_2 V_{\zeta_2}] + [\eta_3 W_{\zeta_3}] + [\eta_4 T_{\zeta_4}],\\
\Omega_{zy} &= [\zeta_1 U_{\eta_1}] + [\zeta_2 V_{\eta_2}] + [\zeta_3 W_{\eta_3}] + [\zeta_4 T_{\eta_4}],
\end{aligned}$$

where

$$U_1 = a_{200}\xi_1^2 + \dots, \qquad V_2 = b_{200}\xi_2^2 + \dots,$$
$$W_3 = c_{200}\xi_3^2 + \dots, \qquad T_4 = d_{200}\xi_4^2 + \dots,$$
$$U_{\eta_1} = \frac{\partial}{\partial \eta_1} U_1, \qquad V_{\eta_2} = \frac{\partial}{\partial \eta_2} V_2, \dots,$$

and after expansion of the expressions $[\xi_1 U_{\eta_1}], \dots, \xi_1^p \eta_1^q \zeta_1^r$ is replaced by $\dfrac{\partial}{\partial a_{pqr}}$,

$$\xi_2^p \eta_2^q \zeta_2^r \text{ by } \frac{\partial}{\partial b_{pqr}}, \quad \xi_3^p \eta_3^q \zeta_3^r \text{ by } \frac{\partial}{\partial c_{pqr}}, \quad \xi_4^p \eta_4^q \zeta_4^r \text{ by } \frac{\partial}{\partial d_{pqr}}.$$

The four components of each operator are independent of one another and therefore commutative with one another; but as in § 10 the $\Omega$'s are not all commutative. In fact, applying the results of § 10 to corresponding pairs of the partial operators, we find the alternants of various pairs of $\Omega$'s to be

$$\begin{aligned}
\Omega_{xy}\Omega_{yx} - \Omega_{yx}\Omega_{xy} &= -[\xi_1 U_{\xi_1}] - [\xi_2 V_{\xi_2}] - [\xi_3 W_{\xi_3}] - [\xi_4 T_{\xi_4}] + [\eta_1 U_{\eta_1}] + [\eta_2 V_{\eta_2}] + [\eta_3 W_{\eta_3}] + [\eta_4 T_{\eta_4}],\\
\Omega_{xy}\Omega_{xz} - \Omega_{xz}\Omega_{xy} &= 0,\\
\Omega_{xy}\Omega_{zx} - \Omega_{zx}\Omega_{xy} &= \Omega_{zy},\\
\Omega_{xy}\Omega_{yz} - \Omega_{yz}\Omega_{xy} &= -\Omega_{xz},\\
\Omega_{xy}\Omega_{zy} - \Omega_{zy}\Omega_{xy} &= 0.
\end{aligned}$$

From these relations deductions can be made similar to those in § 10.

We next write

$$\Omega_1 = \frac{A'}{A}\Omega_{xy} + \frac{A''}{A}\Omega_{xz},$$

$$\Omega_2 = \frac{AB}{Jc''}\Omega_{yx} - \frac{c'}{c''}\Omega_{yz},$$

$$\Omega_3 = -\frac{a''A}{J}\Omega_{zx} - \frac{b''c''}{A}\Omega_{zy}.$$

Then
$$\frac{D_1^l D_2^m D_3^n t}{l!\ m!\ n!} = \frac{A^{l-m} c''^{m-n}}{J^l} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} d_{lmn},$$
$$\frac{D_1^l D_2^m D_3^n u}{l!\ m!\ n!} = \frac{A^{l-m} c''^{m-n}}{J^l} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} a_{lmn}, \ldots$$

Therefore
$$D_{lmn} = \frac{A^{l-m} c''^{m-n}}{J^{l+1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} \begin{vmatrix} d_{lmn} & a_{lmn} & b_{lmn} & c_{lmn} \\ d & a & b & c \\ d' & a' & b' & c' \\ d'' & a'' & b'' & c'' \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots(27).$$

Now write
$$J_1 = \begin{vmatrix} d & b & c \\ d' & b' & c' \\ d'' & b'' & c'' \end{vmatrix} = Ad + A'd' + A''d'',$$
$$J_2 = \begin{vmatrix} a & d & c \\ a' & d' & c' \\ a'' & d'' & c'' \end{vmatrix} = Bd + B'd' + B''d'',$$
$$J_3 = \begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix} = Cd + C'd' + C''d'';$$

therefore
$$D_{lmn} = \frac{A^{l-m} c''^{m-n}}{J^{l+1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} (Jd_{lmn} - J_1 a_{lmn} - J_2 b_{lmn} - J_3 c_{lmn}) \ldots\ldots\ldots\ldots\ldots(28),$$

and as in § 11, if $F$ denotes a homogeneous function of degree $i$, which is also isobaric of partial weights $p_1$, $p_2$, $p_3$,
$$F(D_{lmn}, \ldots) = \frac{A^{p_1-p_2} c''^{p_2-p_3}}{J^{i+p_1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} F(Jd_{lmn} - J_1 a_{lmn} - J_2 b_{lmn} - J_3 c_{lmn}, \ldots) \quad \ldots(29),$$

or using the operators $[U_4]$, $[V_4]$, $[W_4]$ defined in § 14
$$F(D_{lmn}, \ldots) = \frac{A^{p_1-p_2} c''^{p_2-p_3}}{J^{p_1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} e^{-\frac{J_1}{J}[U_4]} e^{-\frac{J_2}{J}[V_4]} e^{-\frac{J_3}{J}[W_4]} F(d_{lmn}, \ldots) \quad \ldots\ldots\ldots(30).$$

This last form does not require $F$ to be homogeneous, though it must be isobaric.

§ 16. If in (27) we put $t = x$, $y$, $z$ in succession we obtain formulae for the interchange of the dependent and independent variables. Write
$$A_{lmn} = \frac{1}{l!\ m!\ n!} \frac{\partial^{l+m+n} x}{\partial u^l \partial v^m \partial w^n},$$
$$B_{lmn} = \frac{1}{l!\ m!\ n!} \frac{\partial^{l+m+n} y}{\partial u^l \partial v^m \partial w^n},$$
$$C_{lmn} = \frac{1}{l!\ m!\ n!} \frac{\partial^{l+m+n} z}{\partial u^l \partial v^m \partial w^n}.$$

Then, provided $l+m+n>1$,

$$A_{lmn} = -\frac{A^{l-m}C''^{m-n}}{J^{l+1}} e^{-[U,\,V,\,W]} \,.\, e^{\Omega_1} \,.\, e^{\Omega_2} \,.\, e^{\Omega_3} (Aa_{lmn} + Bb_{lmn} + Cc_{lmn})$$

$$= -\frac{A^{l-m}C''^{m-n}}{J^{l+1}} e^{-[U,\,V,\,W]} \,.\, e^{\Omega_1} \,.\, e^{\Omega_2} \,.\, e^{\Omega_3} \begin{vmatrix} a_{lmn} & b_{lmn} & c_{lmn} \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \quad \text{.....(31}a\text{)},$$

$$B_{lmn} = -\frac{A^{l-m}C''^{m-n}}{J^{l+1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} (A'a_{lmn} + B'b_{lmn} + C'c_{lmn})$$

$$= -\frac{A^{l-m}C''^{m-n}}{J^{l+1}} e^{-[U,\,V,\,W]} e^{\Omega_1} \,.\, e^{\Omega_2} \,.\, e^{\Omega_3} \begin{vmatrix} a & b & c \\ a_{lmn} & b_{lmn} & c_{lmn} \\ a'' & b'' & c'' \end{vmatrix} \quad \text{.....(31}b\text{)},$$

$$C_{lmn} = -\frac{A^{l-m}C''^{m-n}}{J^{l+1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} (A''a_{lmn} + B''b_{lmn} + C''c_{lmn})$$

$$= -\frac{A^{l-m}C''^{m-n}}{J^{l+1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3} \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a_{lmn} & b_{lmn} & c_{lmn} \end{vmatrix} \quad \text{.....(31}c\text{)}.$$

And if $F(D_{lmn},\, A_{lmn},\, B_{lmn},\, C_{lmn},\, \ldots)$ is a function homogeneous of degree $i$ in $A_{lmn},\, B_{lmn},\, C_{lmn},\, D_{lmn},\, \ldots$ and isobaric of weights $p_1,\, p_2,\, p_3$ in first, second and third suffixes, we have

$$F(D_{lmn},\, A_{lmn},\, B_{lmn},\, C_{lmn},\, \ldots) = (-1)^i \frac{A^{p_1-p_2}C''^{p_2-p_3}}{J^{i+p_1}} e^{-[U,\,V,\,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3}$$

$$.\, F(J_1 a_{lmn} + J_2 b_{lmn} + J_3 c_{lmn} - J d_{lmn},\; Aa_{lmn} + Bb_{lmn} + Cc_{lmn},$$

$$A'a_{lmn} + B'b_{lmn} + C'c_{lmn},\; A''a_{lmn} + B''b_{lmn} + C''c_{lmn},\, \ldots) \quad \text{.....(32)}.$$

Since there are no $d$'s occurring in equations (31$a$), (31$b$), (31$c$) the operators occurring in these equations, but not in (32), may be simplified by the omission of differential operators which affect only $d$'s. Thus $[U_4 T_{\xi_4}]$ may be omitted from $[U,\, x]$, $[\xi_4 T_{\eta_4}]$ from $\Omega_{xy}, \ldots$.

It will be noted that $A_{lmn},\, B_{lmn},\, C_{lmn}$ are the coefficients of $\upsilon^l \nu^m \omega^n$ in the expansions of $\xi,\, \eta,\, \zeta$ when the series

$$\upsilon = a_{100}\xi + \ldots + a_{lmn}\xi^l \eta^m \zeta^n + \ldots,$$

$$\nu = b_{100}\xi + \ldots + b_{lmn}\xi^l \eta^m \zeta^n + \ldots,$$

$$\omega = c_{100}\xi + \ldots + c_{lmn}\xi^l \eta^m \zeta^n + \ldots,$$

are reversed, and $\xi,\, \eta,\, \zeta$ expanded in powers of $\upsilon,\, \nu,\, \omega$.

§ 17. The formulae of § 15 may be adapted so as to give a symbolical form for the differential coefficient of an implicit function. The method is applicable to any number of variables, but for the sake of brevity the work will here be restricted to the case when there is only one dependent and one independent variable.

As in § 16, let

$$u = F(x, y), \quad v = G(x, y);$$

then formula (31*b*) shows how to determine $B_{0m}$ or $\frac{1}{m!}\frac{\partial^m y}{\partial v^m}$; this differential coefficient is obtained on the assumption that $u$ is constant, so that if we take $G(x, y) \equiv x$, we shall obtain $\frac{1}{m!}\frac{d^m y}{dx^m}$ on the assumption that $x$, $y$ are connected by the equation $F(x, y) = \text{const.}$ As in § 16 let $a_{pq}$ stand for

$$\frac{1}{p!\,q!}\frac{\partial^{p+q} F}{\partial x^p \partial y^q};$$

the $b$'s of § 16 are in this case all zero except $b_{10}$ which is equal to unity. Now we have

$$J = \begin{vmatrix} a_{10} & a_{01} \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} a & a' \\ 1 & 0 \end{vmatrix} = -a_{01}$$

so that $A = 0$, and therefore the forms of $\Omega_1$, $\Omega_2$ used in (27) are not applicable. Instead of these forms we may use forms similar to those given in (25*a*, *b*) and obtain

$$K = 1, \quad \Omega_1 = 0, \quad \Omega_2 = -\frac{a_{10}}{a_{01}}\Omega_{xy}.$$

We have therefore, if $m > 1$,

$$B_{0m} = \frac{1}{a_{01}} e^{-[U, V]} e^{-\frac{a_{10}}{a_{01}}\Omega_{xy}} \begin{vmatrix} a_{10} & 1 \\ a_{m0} & b_{m0} \end{vmatrix}.$$

Now, in general, when dealing with special values of the letters, it is necessary to carry out all the operations indicated and then substitute the special values. But in the present case, where all the $b$'s involved in the operators are zero, it is allowable to suppress in the operators all terms which involve $b$'s; for it is obvious from the form of the operators that they never diminish the degree of any function in $b$'s, though they may increase the degree. It therefore follows that the terms which arise from the $b$-parts of the operators will all be zero. We therefore have

$$[U, V] = \frac{A'}{J}[U, y] = \frac{1}{a_{01}}[FF_\eta]$$

$$= \frac{1}{a_{01}}[(a_{20}\xi^2 + a_{11}\xi\eta + a_{02}\eta^2 + a_{30}\xi^3 + \ldots)(a_{11}\xi + 2a_{02}\eta + a_{21}\xi^2 + \ldots)],$$

$$\Omega_{xy} = [\xi F_\eta] = [\xi(a_{11}\xi + 2a_{02}\eta + a_{21}\xi^2 + \ldots)],$$

on the usual understanding that $\xi^p\eta^q$ is replaced by $\dfrac{\partial}{\partial a_{pq}}$. Hence finally the value of $\dfrac{d^m y}{dx^m}$ as found from the equation $F(x, y)=0$ is given by

$$B_{0m}=\frac{1}{m!}\frac{d^m y}{dx^m}=-\frac{1}{a_{01}}e^{-\frac{1}{a_{01}}[FF_\eta]}\,.\,e^{-\frac{a_{10}}{a_{01}}[\xi F_\eta]}\,.\,a_{m0} \quad\text{.....(33).}$$

§ 18. The determination of the differential coefficients of implicit functions is equivalent to the solution of equations by series, so that the method of the last section leads to a symbolical form for the solution of a set of equations of infinite degree. It will be sufficient to illustrate the method by considering the case of a single equation,

$$0=F(x, y)=a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2+a_{30}x^3+\dots;$$

it is required to determine that value of $y$ which vanishes when $x$ vanishes. The solution is

$$y=B_{01}x+B_{02}x^2+\dots$$

where $B_{01}=-\dfrac{a_{10}}{a_{01}}$, and $B_{0m}$ is given by (33). Now let $P$ denote the terms of $F(x, y)$ which are independent of $y$; then the required solution of the equation $F(x, y)=0$ may be written

$$y=-\frac{1}{a_{01}}e^{-\frac{1}{a_{01}}[FF_\eta]}\,.\,e^{-\frac{a_{10}}{a_{01}}[\xi F_\eta]}P \quad\text{.....(34),}$$

where the operators $[FF_\eta]$, $[\xi F_\eta]$ are the same as in the last section. For an equation of finite degree $n$ it is necessary to suppose all the operations carried out, and then all the coefficients $a_{pq}$ for which $p+q>n$ must be made zero.

If $f$ denotes any rational integral function

$$f(y)=e^{-\frac{1}{a_{01}}[FF_\eta]}\,.\,e^{-\frac{a_{10}}{a_{01}}[\xi F_\eta]}f\left(-\frac{P}{a_{01}}\right) \quad\text{.....(34a).}$$

§ 19. As an illustration of the general methods established, we will employ them to effect the change when the variables are linearly transformed. Let the scheme of transformation be

$$X=\alpha x+\alpha' y+\alpha'' z+\alpha''' t,$$
$$Y=\beta x+\beta' y+\beta'' z+\beta''' t,$$
$$Z=\gamma x+\gamma' y+\gamma'' z+\gamma''' t,$$
$$T=\delta x+\delta' y+\delta'' z+\delta''' t,$$

and $T$ being regarded as the dependent variables. Let

$$d_{lmn}=\frac{1}{l!\,m!\,n!}\frac{\partial^{l+m+n}t}{\partial x^l\partial y^m\partial z^n},$$

$$D_{lmn}=\frac{1}{l!\,m!\,n!}\frac{\partial^{l+m+n}T}{\partial X^l\partial Y^m\partial Z^n}.$$

It is required to express $D_{lmn}$ in terms of $d_{lmn}, d_{l'm'n'}, \ldots$. In the formulae $T, X, Y, Z$ are to be written for $t, u, v, w$ so that $a, b, c, \ldots$ will denote first differential coefficients of $T$, not $t$, and therefore

$$a = \alpha + \alpha''' t_x, \quad b = \beta + \beta''' t_x, \ldots$$

Now equation (27) shows that by writing $X, Y, Z, T$ for $u, v, w, t$ the value of $D_{lmn}$ can be found in terms of differential coefficients of $X, Y, Z, T$ with respect to $x, y, z$ and therefore expressed in terms of $t_{pqr}, \ldots$. But the operators $[U, x], \ldots$ which produce the expression can be replaced by others involving differential operators $\frac{\partial}{\partial t_{pqr}}, \ldots$.

For the operator $[U, x]$ or $[X, x]$ may be written

$$\Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (XX_x) \frac{\partial}{\partial X_{pqr}} + \Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (XY_x) \frac{\partial}{\partial Y_{pqr}} + \Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (XZ_x) \frac{\partial}{\partial Z_{pqr}}$$

$$+ \Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (XT_x) \frac{\partial}{\partial T_{pqr}},$$

where $p + q + r > 2$.

Now since all first differential coefficients are removed after operation with $\Delta$'s it is obvious that in $\Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (XX_x), \ldots$ $X, Y, Z, T$ may be replaced by $\alpha''' t, \beta''' t, \gamma''' t, \delta''' t$. Moreover, if $p + q + r \nless 2$,

$$X_{pqr} = \alpha''' t_{pqr}, \quad Y_{pqr} = \beta''' t_{pqr}, \quad Z_{pqr} = \gamma''' t_{pqr}, \quad T_{pqr} = \delta''' t_{pqr};$$

and therefore, for operations on a function of $X_{pqr}, Y_{pqr}, Z_{pqr}, T_{pqr}, \ldots$,

$$\frac{\partial}{\partial t_{pqr}} = \alpha''' \frac{\partial}{\partial X_{pqr}} + \beta''' \frac{\partial}{\partial Y_{pqr}} + \gamma''' \frac{\partial}{\partial Z_{pqr}} + \delta''' \frac{\partial}{\partial T_{pqr}}.$$

Also
$$\Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (XX_x) = \alpha'''^2 \Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (tt_x),$$
$$\Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (XY_x) = \alpha''' \beta''' \Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (tt_x), \text{ \&c.}$$

Therefore $[U, x]$ becomes

$$\alpha''' \Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (tt_x) \left[ \alpha''' \frac{\partial}{\partial X_{pqr}} + \beta''' \frac{\partial}{\partial Y_{pqr}} + \gamma''' \frac{\partial}{\partial Z_{pqr}} + \delta''' \frac{\partial}{\partial T_{pqr}} \right]$$
$$= \alpha''' \Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (tt_x) \frac{\partial}{\partial t_{pqr}}.$$

Now denote the operators

$$\Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (tt_x) \frac{\partial}{\partial t_{pqr}}, \quad \Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (tt_y) \frac{\partial}{\partial t_{pqr}}, \quad \Sigma \Delta_x{}^p \Delta_y{}^q \Delta_z{}^r (tt_z) \frac{\partial}{\partial t_{pqr}}$$

by $V_1, V_2, V_3$. When working with $d_{pqr}$ instead of $t_{pqr}$ it will be more convenient to form the operators $V$ by writing

$$V_1 = [\tau\tau_\xi], \quad V_2 = [\tau\tau_\eta], \quad V_3 = [\tau\tau_\zeta],$$

where
$$\tau = d_{200}\xi^2 + d_{002}\eta^2 + \ldots$$

and after the algebraical multiplications $\xi^p \eta^q \zeta^r$ is replaced by $\frac{\partial}{\partial d_{pqr}}$.

We shall then find that $[U, x]$, $[U, y]$, $[U, z]$, $[V, x]$,... become $\alpha''' V_1$, $\alpha''' V_2$, $\alpha''' V_3$, $\beta''' V_1, \ldots$; and it finally appears that $[U, V, W]$ becomes

$$\frac{A\alpha''' + B\beta''' + C\gamma'''}{J} V_1 + \frac{A'\alpha''' + B'\beta''' + C'\gamma'''}{J} V_2 + \frac{A''\alpha''' + B''\beta''' + C''\gamma'''}{J} V_3 = V, \text{ say,}$$

so that

$$V = \frac{1}{J} \begin{vmatrix} V_1 & V_2 & V_3 & 0 \\ a & a' & a'' & \alpha''' \\ b & b' & b'' & \beta''' \\ c & c' & c'' & \gamma''' \end{vmatrix}$$

$$= \frac{1}{J} \begin{vmatrix} V_1 & V_2 & V_3 & 0 \\ \alpha + \alpha'' t_x, & \alpha' + \alpha''' t_y, & \alpha'' + \alpha''' t_z, & \alpha''' \\ \beta + \beta''' t_x, & \beta' + \beta''' t_y, & \beta'' + \beta''' t_z, & \beta''' \\ \gamma + \gamma''' t_x, & \gamma' + \gamma''' t_y, & \gamma'' + \gamma''' t_z, & \gamma''' \end{vmatrix}$$

$$= \frac{1}{J} \begin{vmatrix} V_1 & V_2 & V_3 & 0 \\ \alpha & \alpha' & \alpha'' & \alpha''' \\ \beta & \beta' & \beta'' & \beta''' \\ \gamma & \gamma' & \gamma'' & \gamma''' \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35).$$

When there are $n$ independent variables the corresponding formula for $V$ is

$$V = \frac{(-1)^{n-1}}{J} \begin{vmatrix} V_1, & V_2, \ldots V_n, & 0 \\ \alpha, & \alpha', \ldots \alpha^{(n-1)}, & \alpha^{(n)} \\ \beta, & \beta', \ldots \beta^{(n-1)}, & \beta^{(n)} \\ \gamma, & \gamma', \ldots \gamma^{(n-1)}, & \gamma^{(n)} \\ \ldots & \ldots & \ldots \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36).$$

The formula (26) then gives

$$D_{lmn} = \frac{A^{l-m} C''^{m-n}}{J^{l+1}} e^{-V} \begin{vmatrix} D_1^l D_2^m D_3^n T, & D_1^l D_2^m D_3^n X, & D_1^l D_2^m D_3^n Y, & D_1^l D_2^m D_3^n Z \\ T_x & X_x & Y_x & Z_x \\ T_y & X_y & Y_y & Z_y \\ T_z & X_z & Y_z & Z_z \end{vmatrix} \quad \ldots\ldots (37).$$

Now, $l + m + n$ being greater than unity, the determinant becomes

$$\begin{vmatrix} \delta''' & \alpha''' & \beta''' & \gamma''' \\ \delta + \delta''' t_x & \alpha + \alpha''' t_x & \beta + \beta''' t_x & \gamma + \gamma''' t_x \\ \delta' + \delta''' t_y & \alpha' + \alpha''' t_y & \beta' + \beta''' t_y & \gamma' + \gamma''' t_y \\ \delta'' + \delta''' t_z & \alpha'' + \alpha''' t_z & \beta'' + \beta''' t_z & \gamma'' + \gamma''' t_z \end{vmatrix} D_1^l D_2^m D_3^n t$$

$$= \begin{vmatrix} \delta''' & \alpha''' & \beta''' & \gamma''' \\ \delta & \alpha & \beta & \gamma \\ \delta' & \alpha' & \beta' & \gamma' \\ \delta'' & \alpha'' & \beta'' & \gamma'' \end{vmatrix} D_1^l D_2^m D_3^n t$$

$$= \begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \\ \alpha'' & \beta'' & \gamma'' & \delta'' \\ \alpha''' & \beta''' & \gamma''' & \delta''' \end{vmatrix} D_1^l D_2^m D_3^n t$$

$$= M \,.\, D_1^l D_2^m D_3^n t, \text{ say,}$$

so that $M$ is the modulus of the linear transformation.

To transform $D_1^l D_2^m D_3^n t$ we use the operators

$$\omega_{xy} = [\xi \tau_\eta], \qquad \omega_{xz} = [\xi \tau_\zeta],$$

$$\omega_{yx} = [\eta \tau_\xi], \qquad \omega_{yz} = [\eta \tau_\zeta],$$

$$\omega_{zx} = [\zeta \tau_\xi], \qquad \omega_{zy} = [\zeta \tau_\eta],$$

where $\tau = d_{200}\xi^2 + d_{002}\eta^2 + \dots$, and the operators are formed in the usual way by replacing $\xi^p \eta^q \zeta^r$ with $\dfrac{\partial}{\partial d_{pqr}}$. The properties of these $\omega$'s are precisely similar to those of the $\omega$'s investigated in § 10, and just as in previous cases we find that if we write

$$\Omega_1 = \frac{A'}{A}\,\omega_{xy} + \frac{A''}{A}\,\omega_{xz},$$

$$\Omega_2 = \frac{AB}{Jc''}\,\omega_{yx} - \frac{c'}{c''}\,\omega_{yz},$$

$$\Omega_3 = -\frac{a''A}{J}\,\omega_{zx} - \frac{b''c''}{A}\,\omega_{zy},$$

we shall have finally

$$D_{lmn} = M\,\frac{A^{l-m}c''^{m-n}}{J^{l+1}}\,e^{-V}e^{\Omega_1}e^{\Omega_2}e^{\Omega_3}\,d_{lmn} \quad \dots\dots\dots\dots\dots\dots\dots\dots (38).$$

And if $F(d_{lmn},\ d_{l'm'n'}, \dots)$ is a pure homogeneous function of degree $i$ and isobaric of weights $p_1$, $p_2$, $p_3$ in first, second and third suffixes,

$$F(D_{lmn},\ D_{l'm'n'}, \dots) = \frac{M^i}{J^{i+p_1}}\,A^{p_1-p_2}c''^{p_2-p_3}e^{-V}e^{\Omega_1}e^{\Omega_2}e^{\Omega_3}\,F(d_{lmn},\ d_{l'm'n'}, \dots) \quad \dots\dots (39).$$

The value of $J$ is

$$\begin{vmatrix} \alpha + \alpha'''d, & \alpha' + \alpha'''d', & \alpha'' + \alpha'''d'' \\ \beta + \beta'''d, & \beta' + \beta'''d', & \beta'' + \beta'''d'' \\ \gamma + \gamma'''d, & \gamma' + \gamma'''d', & \gamma'' + \gamma'''d'' \end{vmatrix}$$

and $A$, $B$, $C$, ... are the minors of this determinant. It will be noticed that products of $d$, $d'$, $d''$ do not appear in $J$, which is therefore a linear function of these quantities. In fact another form is

$$J = \begin{vmatrix} \alpha, & \alpha', & \alpha'', & \alpha''' \\ \beta, & \beta', & \beta'', & \beta''' \\ \gamma, & \gamma', & \gamma'', & \gamma''' \\ -d, & -d', & -d'', & 1 \end{vmatrix}$$

$$= \Delta''' - \Delta d - \Delta' d' - \Delta'' d'' \quad \text{.....(40)},$$

where $\Delta$, $\Delta'$, $\Delta''$, $\Delta'''$ are the minors of $\delta$, $\delta'$, $\delta''$, $\delta'''$ in the determinant $M$.

§ 20. As in § 12 the part of the operator which depends on the $\omega$'s may be expressed in different forms.

For instance in the case of two independent variables we have

$$F(D_{lm}, \ldots) = Ke^{-V}e^{\Omega_1}e^{\Omega_2}F(d_{lm}, \ldots) \quad \text{.....(41)},$$

where

$$V = -\frac{1}{J}\begin{vmatrix} V_1 & V_2 & 0 \\ \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{vmatrix} \quad \text{.....(41a)},$$

and $K$, $\Omega_1$, $\Omega_2$ may have either of the two sets of values

$$K = \frac{M^i A^{p_1-p_2}}{J^{p_1+i}}, \quad \Omega_1 = \frac{A'}{A}\omega_{xy}, \quad \Omega_2 = \frac{AB}{J}\omega_{yx} \quad \text{.....(41b)};$$

$$K = \frac{M^i B'^{p_2-p_1}}{J^{p_2+i}}, \quad \Omega_1 = \frac{B}{B'}\omega_{yx}, \quad \Omega_2 = \frac{A'B'}{J}\omega_{xy} \quad \text{.....(41c)}.$$

And, equally well,

$$F(D_{lm}, \ldots) = Ke^{-V}e^{\Omega_1}e^{\Omega_2}F(d_{ml}, \ldots) \quad \text{.....(41d)},$$

where $V$ is as before, and $K$, $\Omega_1$, $\Omega_2$ may have either of the two sets of values

$$K = (-1)^{p_2}\frac{M^i A'^{p_1-p_2}}{J^{p_1+i}}, \quad \Omega_1 = \frac{A}{A'}\omega_{yx}, \quad \Omega_2 = -\frac{A'B'}{J}\omega_{xy} \quad \text{.....(41e)};$$

$$K = (-1)^{p_1}\frac{M^i B^{p_2-p_1}}{J^{p_2+i}}, \quad \Omega_1 = \frac{B'}{B}\omega_{xy}, \quad \Omega_2 = -\frac{AB}{J}\omega_{yx} \quad \text{.....(41f)}.$$

For example, suppose it is required to change the variables cyclically so that $t$, $x$, $y$ are changed to $x$, $y$, $t$ and $x$ is the new dependent variable. Let $x_{lm}$ stand for $\dfrac{1}{l!\,m!}\dfrac{\partial^{l+m}x}{\partial y^l \partial t^m}$.

The scheme of transformation is

$$X = 0 \,.\, x + 1 \,.\, y + 0 \,.\, t,$$
$$Y = 0 \,.\, x + 0 \,.\, y + 1 \,.\, t,$$
$$T = 1 \,.\, x + 0 \,.\, y + 0 \,.\, t.$$

Therefore
$$J=\begin{vmatrix} 0 & 1 \\ t_x & t_y \end{vmatrix}=-t_x,\quad M=1,$$

$$V=-\frac{1}{J}\begin{vmatrix} V_1 & V_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=\frac{V_1}{t_x}.$$

Here $B'=0$ and therefore the form (41$c$) is not applicable, but the form (41$b$) gives

$$F(x_{lm},\ldots)=\frac{t_y^{p_1-p_2}}{(-t_x)^{p_1+i}}e^{-\frac{V_1}{t_x}}e^{-\frac{t_x}{t_y}\omega_{xy}}e^{\frac{t_y}{t_x}\omega_{yx}}F(d_{lm},\ldots)\quad\ldots\ldots\ldots\ldots\ldots\ldots(42a);$$

and the forms (41$e$), (41$f$) give identical results, viz.,

$$F(x_{lm},\ldots)=(-1)^{p_2}\frac{(-t_x)^{p_1-p_2}}{(-t_x)^{i+p_1}}e^{-\frac{V_1}{t_x}}e^{-\frac{t_y}{t_x}\omega_{yx}}F(d_{ml},\ldots)$$

$$=(-1)^i\frac{1}{t_x^{i+p_2}}e^{-\frac{V_1}{t_x}}e^{-\frac{t_y}{t_x}\omega_{yx}}F(d_{ml},\ldots)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42b).$$

If it is required to make the second cyclical change from $t$, $x$, $y$ to $y$, $t$, $x$ so that $y$ is the dependent variable, let $y_{lm}=\frac{1}{l!\,m!}\frac{\partial^{l+m}y}{\partial t^l\partial x^m}$. Then the scheme of transformation is

$$X=0.x+0.y+1.t,$$
$$Y=1.x+0.y+0.t,$$
$$T=0.x+1.y+0.t.$$

Therefore
$$J=\begin{vmatrix} t_x & t_y \\ 1 & 0 \end{vmatrix}=-t_y,$$
$$M=1,$$
$$V=-\frac{1}{J}\begin{vmatrix} V_1 & V_2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}=\frac{V_2}{t_y}.$$

Here $A=0$, and the form (41$b$) is not applicable, but the other forms give

$$F(y_{lm},\ldots)=\frac{(t_x)^{p_2-p_1}}{(-t_y)^{p_2+i}}e^{-\frac{V_2}{t_y}}e^{-\frac{t_y}{t_x}\omega_{yx}}e^{\frac{t_x}{t_y}\omega_{xy}}F(d_{lm},\ldots)\ldots\ldots\ldots\ldots\ldots\ldots(43a)$$

and

$$F(y_{lm},\ldots)=(-1)^{p_2}\frac{(-1)^{p_1-p_2}}{(-t_y)^{i+p_1}}e^{-\frac{V_2}{t_y}}e^{-\frac{t_x}{t_y}\omega_{xy}}F(d_{ml},\ldots)$$

$$=(-1)^i\frac{1}{t_y^{i+p_1}}e^{-\frac{V_2}{t_y}}e^{-\frac{t_x}{t_y}\omega_{xy}}F(d_{ml},\ldots)$$

$$=(-1)^i\frac{1}{t_{01}^{i+p_1}}e^{-\frac{V_2}{t_{01}}}e^{-\frac{t_{10}}{t_{01}}\omega_{xy}}F(d_{ml},\ldots)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43b).$$

§ 21. The alternants of the operators $\omega$ with one another have been already examined in § 10; it remains to examine the alternants of the operators $V$ combined with $\omega$'s. These alternants have, in the case of two independent variables, been given by Prof. Elliott (*Proc. Lond. Math. Soc.*, Vol. XIX. p. 9); but the proof is much simplified by making full use of the symbolical form for the operators. Only typical cases sufficient to establish the general results will be considered. We have

$$V_1 = [TT_\xi], \quad \omega_{xy} = [\xi T_\eta], \quad \omega_{yx} = [\eta T_\xi], \quad \omega_{yz} = [\eta T_\zeta],$$

where $T = d_{200}\xi^2 + \ldots$, and after multiplication $\xi^p \eta^q \zeta^r$ is replaced by $\dfrac{\partial}{\partial d_{pqr}}$.

Therefore
$$\begin{aligned} V_1 V_2 - V_2 V_1 &= [TT_\xi][TT_\eta] - [TT_\eta][TT_\xi] \\ &= \left[TT_\xi T_\eta + T\frac{\partial}{\partial \eta}(TT_\xi)\right] - \left[TT_\eta T_\xi + T\frac{\partial}{\partial \xi}(TT_\eta)\right] \\ &= 0 \quad \ldots\ldots (44a), \end{aligned}$$

$$\begin{aligned} V_1\omega_{xy} - \omega_{xy}V_1 &= [TT_\xi][\xi T_\eta] - [\xi T_\eta][TT_\xi] \\ &= \left[\xi\frac{\partial}{\partial \eta}(TT_\xi)\right] - \left[\xi T_\eta T_\xi + T\frac{\partial}{\partial \xi}(\xi T_\eta)\right] \\ &= -[TT_\eta] = -V_2 \quad \ldots\ldots (44b), \end{aligned}$$

$$\begin{aligned} V_1\omega_{yx} - \omega_{yx}V_1 &= [TT_\xi][\eta T_\xi] - [\eta T_\xi][TT_\xi] \\ &= \left[\eta\frac{\partial}{\partial \xi}(TT_\xi)\right] - \left[\eta T_\xi T_\xi + T\frac{\partial}{\partial \xi}(\eta T_\xi)\right] \\ &= 0 \quad \ldots\ldots (44c), \end{aligned}$$

$$\begin{aligned} V_1\omega_{yz} - \omega_{yz}V_1 &= [TT_\xi][\eta T_\zeta] - [\eta T_\zeta][TT_\xi] \\ &= \left[\eta\frac{\partial}{\partial \zeta}(TT_\xi)\right] - \left[\eta T_\zeta T_\xi + T\frac{\partial}{\partial \xi}(\eta T_\zeta)\right] \\ &= 0 \quad \ldots\ldots (44d). \end{aligned}$$

Similarly
$$V_1\omega_{xz} - \omega_{xz}V_1 = -V_3 \quad \ldots\ldots (44e),$$

and generally $V_1$ or $[TT_\xi]$ is commutative with all $\omega$'s except those which have $x$ for the first suffix, whilst all the $V$'s are commutative with one another.

§ 22. The applications to the theory of pure cyclicants are easily made. A cyclicant is defined as a function of differential coefficients which is unaltered when the dependent and independent variables are interchanged in any way whatever, save for the introduction of a factor which involves first differential coefficients only. The cyclicant is pure if it involves no first differential coefficients.

In the case of three independent variables, and the method will be perfectly general for any number, we shall show that if the function, supposed pure, is invariable

save for the factor mentioned above, when $t$ is interchanged with $x$ and $y$, $z$ are unaltered, and also unaltered when $t$ is interchanged with $y$ and $z$, $x$ are unaltered, and also when $t$ is interchanged with $z$ and $x$, $y$ are unaltered, then the function is invariable when any interchange whatever is made, and moreover the function is invariable when the general linear transformation is applied. We shall find that the necessary and sufficient conditions for the invariance of a homogeneous and isobaric function are that it be annihilated by the three operators $V_1$, $V_2$, $V_3$ and the six operators $\omega_{xy}$, $\omega_{xz}$, $\omega_{yx}$, $\omega_{yz}$, $\omega_{zx}$, $\omega_{zy}$. These conditions, though necessary and sufficient, are not independent. For it is evident, as in § 10, that annihilation by three $\omega$'s such as $\omega_{yz}$, $\omega_{zx}$, $\omega_{xy}$ will ensure annihilation by the remaining $\omega$'s, and it is proved in § 21, that annihilation by the $\omega$'s and $V_1$ will ensure annihilation by $V_2$ and $V_3$. Now annihilation by the $\omega$'s implies that the function is invariable when the independent variables only are linearly transformed, so that a pure function will be a cyclicant if it is unaltered by linear transformation of the independent variables and unaltered also by the interchange of the dependent and one independent variable.

In consequence of annihilation by the $\omega$'s any pure cyclicant will be an invariant of the system of quantics in $\xi$, $\eta$, $\zeta$,

$$\begin{aligned}
&d_{200}\xi^2 + d_{020}\eta^2 + d_{002}\zeta^2 + d_{110}\xi\eta + d_{101}\xi\zeta + d_{011}\eta\zeta,\\
&d_{300}\xi^3 + \ldots + d_{211}\xi^2\eta + \ldots + d_{111}\xi\eta\zeta,\\
&d_{400}\xi^4 + \ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

and conversely any invariant of these quantics which is annihilated by $V_1$ will be a pure cyclicant.

When the number of independent variables is $n$, there will be $n$ operators of the $V$ type and $n(n-1)$ operators of the $\omega$ type.

§ 23. To make the transformation by interchange of $t$ and $x$ the scheme will be

$$\begin{aligned}
X &= 0\,.\,x + 0\,.\,y + 0\,.\,z + 1\,.\,t,\\
Y &= 0\,.\,x + 1\,.\,y + 0\,.\,z + 0\,.\,t,\\
Z &= 0\,.\,x + 0\,.\,y + 1\,.\,z + 0\,.\,t,\\
T &= 1\,.\,x + 0\,.\,y + 0\,.\,z + 0\,.\,t,
\end{aligned}$$

and the meaning of $D_{lmn}$ will be $\dfrac{1}{l!\,m!\,n!}\,\dfrac{\partial^{l+m+n}x}{\partial t^l \partial y^m \partial z^n}$.

Here $M = -1$,

$$J = \begin{vmatrix} t_x & t_y & t_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = t_x,$$

$$A = 1,\quad A' = 0,\quad A'' = 0,\quad B = -t_y,\quad c' = 0,\quad a'' = t_z,\quad b'' = 0,\quad c'' = 1.$$

Therefore $$\Omega_1 = 0, \quad \Omega_2 = -\frac{t_y}{t_x}\omega_{yx}, \quad \Omega_3 = -\frac{t_z}{t_x}\omega_{zx},$$

and $$V = \begin{vmatrix} V_1 & V_2 & V_3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = V_1.$$

Therefore if $F(d_{lmn}, d_{l'm'n'}, \ldots)$ is a pure homogeneous function of degree $i$ and isobaric of weights $p_1$, $p_2$, $p_3$ in first, second and third suffixes,

$$F(D_{lmn}, D_{l'm'n'}, \ldots) = \frac{(-1)^i}{t_x^{i+p_1}} e^{-\frac{V_1}{t_x}} e^{-\frac{t_y}{t_x}\omega_{yx}} e^{-\frac{t_z}{t_x}\omega_{zx}} F(d_{lmn}, d_{l'm'n'}, \ldots).$$

The right-hand side can be arranged in powers of $\frac{1}{t_x}$, $\frac{t_y}{t_x}$, $\frac{t_z}{t_x}$, and since these are independent quantities it is obvious by observing the coefficients of their lowest powers that, in order that $F(d_{lmn}, \ldots)$ may be invariable, save for a factor, it is necessary that the function be annihilated by $V_1$, $\omega_{yx}$ and $\omega_{zx}$. These conditions are obviously sufficient, and therefore if the conditions are satisfied we have

$$F(D_{lmn}, D_{l'm'n'}, \ldots) = \frac{(-1)^i}{t_x^{i+p_1}} F(d_{lmn}, d_{l'm'n'}, \ldots).$$

Similarly the necessary and sufficient conditions that $F$ may be invariable when $t$ is interchanged with $y$, and $x$, $z$ are unaltered, are that $F$ be annihilated by $V_2$, $\omega_{xy}$ and $\omega_{zy}$; and, when $t$ is interchanged with $z$ and $x$, $y$ are unaltered, the necessary and sufficient conditions for the permanence of $F$ are that it be annihilated by $V_3$, $\omega_{xz}$ and $\omega_{yz}$.

If $F$ is annihilated by all the operators $V$ and $\omega$, equation (39) shows that $F$ will be permanent in form, save for a factor, when any interchanges of variables are made or when both dependent and independent variables are changed by any linear transformation.

Since the annihilation of a function isobaric in first, second and third suffixes by the $\omega$'s implies that the three weights are equal, equation (39) shows that if $F$ be a pure cyclicant the effect of the general linear transformation upon it is to transform it into

$$\frac{M^i}{J^{i+p}} F,$$

where $i$ is the degree of $F$ and $p$ the weight in either set of suffixes.

In order that a function may remain permanent in form when the variables are changed by the general linear transformation it is therefore necessary that it be homogeneous and isobaric in each set of suffixes throughout.

§ 24. As another illustration we will give a proof of a theorem established by Prof. Elliott in a paper "On Pure Ternary Reciprocants, and Functions allied to them" (*Proc. Lond. Math. Soc.* Vol. XIX.). In this paper he considers two independent variables $x$, $y$ with a dependent variable $z$. His operators $V_1$ and $V_2$ are the same as those considered in §§ 19, 20 and 21, with $z$ written instead of $t$; his $\Omega_1$ is $\omega_{yx}$ and his $\Omega_2$ is $\omega_{xy}$; and he writes $z_{pq}$ for $\frac{1}{p!\,q!}\frac{\partial^{p+q}z}{\partial x^p \partial y^q}$. He considers a "reciprocantive covariant"

$$(P_0, P_1, P_2, \dots P_m)(u, v)^m,$$

where $u$, $v$ are any quantities and $P_0, P_1, \dots P_m$ are functions of $z_{pq}, \dots$ where $p+q \nless 2$, such that $P_0$ is a homogeneous and isobaric function annihilated by $V_1$, $V_2$ and $\omega_{yx}$, and

$$\omega_{xy}P_0 = mP_1, \quad \omega_{xy}P_1 = (m-1)P_2, \dots \omega_{xy}P_{m-1} = P_m, \quad \omega_{xy}P_m = 0.$$

In consequence of these conditions the function is a covariant of the emanants

$$(d_{20}, d_{11}, d_{02} \between u, v)^2, \quad (d_{30}, d_{21}, d_{12}, d_{03} \between u, v)^3, \dots$$

Therefore, if $w_1$, $w_2$ are the partial weights of $P_0$, $m = w_1 - w_2$ and the function is only altered by the factor $(-1)^{w_2}$ when $u$, $v$ are interchanged. Hence if in $P_r$ each quantity $z_{rs}$ is replaced by $z_{sr}$ the result is equal to $(-1)^{w_2}P_{m-r}$, and the quantities $P$ therefore satisfy the conditions

$$\omega_{yx}P_m = mP_{m-1}, \quad \omega_{yx}P_{m-1} = (m-1)P_{m-2}, \dots \omega_{yx}P_1 = P_0, \quad \omega_{yx}P_0 = 0.$$

Prof. Elliott shows also that all the $P$'s are annihilated by $V_1$ and $V_2$; this property following from the relations

$$\omega_{yx}V_1 - V_1\omega_{yx} = 0, \qquad \omega_{xy}V_2 - V_2\omega_{xy} = 0,$$
$$\omega_{xy}V_1 - V_1\omega_{xy} = V_2, \quad \omega_{yx}V_2 - V_2\omega_{yx} = V_1.$$

See § 21.

Now let the variables be cyclically transformed from $z$, $x$, $y$ to $x$, $y$, $z$ so that $x$ is the new dependent variable. Let $x_{pq}$ denote $\frac{1}{p!\,q!}\frac{\partial^{p+q}x}{\partial y^p \partial z^q}$, and let $P_0(x)$ denote the result of substituting $x_{pq}, \dots$ for $z_{pq}, \dots$ in $P_0$.

Similarly when the variables are transformed from $z$, $x$, $y$ to $y$, $z$, $x$ so that $y$ is the new dependent variable, let $y_{pq}$ denote $\frac{1}{p!\,q!}\frac{\partial^{p+q}y}{\partial z^p \partial x^q}$, and let $P_m(y)$ denote the result of substituting $y_{pq}, \dots$ for $z_{pq}, \dots$ in $P_m$. Then Prof. Elliott's theorem states that*

$$(-1)^{i+w_2}\frac{P_0(x)}{x_{01}^{\,i+w_1}} = (-1)^{i+w_1}\frac{P_m(y)}{y_{10}^{\,i+w_1}} = (P_0, P_1, P_2, \dots P_m)(-z_{01}, z_{10})^m,$$

where $i$ is the degree of $P_0$.

* Prof. Elliott gives different powers of $(-1)$ in his statement of the theorem, but there is a slight error in his work which accounts for the difference.

The theorem follows at once from the results of § 20. Using equation (42 $a$), we have

$$P_0(x) = \frac{z_y^{w_1-w_2}}{(-z_x)^{i+w_1}} e^{-\frac{V_1}{z_x}} e^{-\frac{z_x}{z_y}\omega_{xy}} e^{\frac{z_y}{z_x}\omega_{yx}} P_0(z).$$

But $\omega_{yx}$ annihilates $P_0(z)$, and $V_1$ annihilates not only $P_0(z)$ but every function of the form $\omega_{xy}{}^r P_0(z)$; therefore

$$\begin{aligned}
P_0(x) &= \frac{z_y^m}{(-z_x)^{i+w_1}} e^{-\frac{z_x}{z_y}\omega_{xy}} P_0(z) \\
&= \frac{z_y^m}{(-z_x)^{i+w_1}} \left[1 - \frac{z_x}{z_y}\omega_{xy} + \frac{1}{2!}\left(\frac{z_x}{z_y}\right)^2 \omega^2{}_{xy} - \dots\right] P_0(z) \\
&= \frac{1}{(-z_x)^{i+w_1}} \left[P_0 z_y^m - mP_1 z_x z_y^{m-1} + \frac{m(m-1)}{1.2} P_2 z_x^2 z_y^{m-2} - \dots + (-1)^m P_m z_x^m\right] \\
&= \frac{(-1)^m}{(-z_x)^{i+w_1}} (P_0, P_1, P_2, \dots P_m)(-z_y, z_x)^m \\
&= \frac{(-1)^{i+w_2}}{z_{10}^{i+w_1}} (P_0, P_1, P_2, \dots P_m)(-z_{01}, z_{10})^m.
\end{aligned}$$

Again, the first and second partial weights of $P_m$ are $w_2$ and $w_1$ respectively; therefore by equation (43 $a$)

$$P_m(y) = \frac{z_x^{w_1-w_2}}{(-z_y)^{i+w_1}} e^{-\frac{V_2}{z_y}} e^{-\frac{z_y}{z_x}\omega_{yx}} e^{\frac{z_x}{z_y}\omega_{xy}} P_m(z).$$

But $\omega_{xy}$ annihilates $P_m$, and $V_2$ annihilates not only $P_m$ but every function of the form $\omega_{yx}{}^r P_m$; therefore

$$\begin{aligned}
P_m(y) &= \frac{z_x^m}{(-z_y)^{i+w_1}} e^{-\frac{z_y}{z_x}\omega_{yx}} P_m(z) \\
&= \frac{z_x^m}{(-z_y)^{i+w_1}} \left[1 - \frac{z_y}{z_x}\omega_{yx} + \frac{1}{2!}\left(\frac{z_y}{z_x}\right)^2 \omega_{yx}{}^2 - \dots + \frac{(-1)^m}{m!}\left(\frac{z_y}{z_x}\right)^m \omega_{yx}{}^m\right] P_m \\
&= \frac{1}{(-z_y)^{i+w_1}} \left[P_0(-z_y)^m + mP_1(-z_y)^{m-1} z_x + \frac{m(m-1)}{1.2} P_2(-z_y)^{m-2} z_x^2 + \dots + P_m z_x^m\right] \\
&= \frac{1}{(-z_y)^{i+w_1}} (P_0, P_1, P_2, \dots P_m)(-z_y, z_x)^m.
\end{aligned}$$

The two results establish the theorem.

§ 25. It has been seen that when the variables are linearly transformed from $t, x, y, z$ to $T, X, Y, Z$ a pure function of differential coefficients will be unaltered in form, save for a factor, provided it is annihilated by the operators $V_1$, $V_2$, $V_3$ and $\omega_{xy}$, $\omega_{xz}$, .... It will be convenient temporarily to denote $V_1$, $V_2$, $V_3$ by $V_x$, $V_y$, $V_z$. It is evident that this permanence of form would be ensured if the transformed function expressed in terms of differential coefficients of $T$ with respect to $X$, $Y$, $Z$

were annihilated by operators $V_X$, $V_Y$, $V_Z$, $\omega_{XY}$, $\omega_{XZ}$, ... formed with the quantities $D_{pqr}$, $A_{pqr}$, ... in the same way as $V_x$, $V_y$, $V_z$, $\omega_{xy}$, $\omega_{xz}$, ... are formed with $d_{pqr}$, $a_{pqr}$, .... It must be possible therefore to express the effect of these operators $V_X$, $V_Y$, $V_Z$, $\omega_{XY}$, $\omega_{XZ}$, ... on the transformed function by means of the original operators acting on the original function. It is now proposed to examine into the manner in which these operators $V_X$, $\omega_{XY}$, ... can be so expressed.

The scheme of transformation is, as before,

$$X = \alpha x + \alpha' y + \alpha'' z + \alpha''' t,$$
$$Y = \beta x + \beta' y + \beta'' z + \beta''' t,$$
$$Z = \gamma x + \gamma' y + \gamma'' z + \gamma''' t,$$
$$T = \delta x + \delta' y + \delta'' z + \delta''' t,$$

and $t$, $T$ are regarded as the respective dependent variables.

Suppose now that $x$, $y$, $z$ receive increments $\xi$, $\eta$, $\zeta$, and let the consequent increments in $t$, $X$, $Y$, $Z$, $T$ be $\tau$, $\xi'$, $\eta'$, $\zeta'$, $\tau'$, so that

$$\tau = d_{100}\xi + d_{010}\eta + d_{001}\zeta + d_{200}\xi^2 + d_{020}\eta^2 + \dots,$$
$$\tau' = D_{100}\xi' + D_{010}\eta' + D_{001}\zeta' + D_{200}\xi'^2 + D_{020}\eta'^2 + \dots.$$

Then
$$V_x = V_1 = (\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta)(\tau_\xi - d_{100}),$$
$$\omega_{xy} = \xi(\tau_\eta - d_{010}),$$
$$\omega_{yx} = \eta(\tau_\xi - d_{100}), \text{ \&c.},$$

where after expansion $\xi^p\eta^q\zeta^r$ is replaced by $\dfrac{\partial}{\partial d_{pqr}}$.

Therefore also

$$V_X = (\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta')\left(\frac{\partial\tau'}{\partial\xi'} - D_{100}\right),$$
$$\omega_{XY} = \xi'\left(\frac{\partial\tau'}{\partial\eta'} - D_{010}\right),$$
$$\omega_{YX} = \eta'\left(\frac{\partial\tau'}{\partial\xi'} - D_{100}\right), \text{ \&c.},$$

when $\xi'^p\eta'^q\zeta'^r$ is replaced after expansion by $\dfrac{\partial}{\partial D_{pqr}}$.

Now let $F(\xi', \eta', \zeta')$ be the symbolical expression for an operator obtained by expanding $F(\xi', \eta', \zeta')$ in powers of $\xi'$, $\eta'$, $\zeta'$ and replacing $\xi'^p\eta'^q\zeta'^r$ by $\dfrac{\partial}{\partial D_{pqr}}$, it being understood that $F$ contains no term for which $p+q+r<2$. Then Prof. Elliott has

proved* that the expression for this operator in terms of $\frac{\partial}{\partial d_{pqr}}$ is obtained by expanding a certain expression in powers of $\xi$, $\eta$, $\zeta$ and replacing $\xi^p\eta^q\zeta^r$ by $\frac{\partial}{\partial d_{pqr}}$. This expression is

$$\frac{1}{M}F(\xi', \eta', \zeta')\left(\Delta''' - \Delta\frac{\partial\tau}{\partial\xi} - \Delta'\frac{\partial\tau}{\partial\eta} - \Delta''\frac{\partial\tau}{\partial\zeta}\right),$$

where
$$M = \begin{vmatrix} \alpha, & \alpha', & \alpha'', & \alpha''' \\ \beta, & \beta', & \beta'', & \beta''' \\ \gamma, & \gamma', & \gamma'', & \gamma''' \\ \delta, & \delta', & \delta'', & \delta''' \end{vmatrix}$$

and $\Delta$, $\Delta'$, $\Delta''$, $\Delta'''$ are the minors of $\delta$, $\delta'$, $\delta''$, $\delta'''$ in $M$.

The application of this rule to the operators considered here is simplified by use of formula (47) which we now proceed to establish. The rule itself may also be deduced from this formula, but Prof. Elliott adopts a different mode of proof.

We have, in consequence of the scheme of transformation,

$$\begin{aligned} \xi' &= \alpha\xi + \alpha'\eta + \alpha''\zeta + \alpha'''\tau, \\ \eta' &= \beta\xi + \beta'\eta + \beta''\zeta + \beta'''\tau, \\ \zeta' &= \gamma\xi + \gamma'\eta + \gamma''\zeta + \gamma'''\tau, \\ \tau' &= \delta\xi + \delta'\eta + \delta''\zeta + \delta'''\tau. \end{aligned} \quad \text{.....(45).}$$

The simplest way of finding $D_{100}$, $D_{010}$, $D_{001}$ is to determine them as the coefficients of $\xi'$, $\eta'$, $\zeta'$ when $\tau'$ is expressed in terms of these quantities.

Now, neglecting higher powers of $\xi$, $\eta$, $\zeta$ than the first, the last set of equations may be written

$$\begin{aligned} \xi' &= (\alpha + \alpha'''d_{100})\,\xi + (\alpha' + \alpha'''d_{010})\,\eta + (\alpha'' + \alpha'''d_{001})\,\zeta, \\ \eta' &= (\beta + \beta'''d_{100})\,\xi + (\beta' + \beta'''d_{010})\,\eta + (\beta'' + \beta'''d_{001})\,\zeta, \\ \zeta' &= (\gamma + \gamma'''d_{100})\,\xi + (\gamma' + \gamma'''d_{010})\,\eta + (\gamma'' + \gamma'''d_{001})\,\zeta, \\ \tau' &= (\delta + \delta'''d_{100})\,\xi + (\delta' + \delta'''d_{010})\,\eta + (\delta'' + \delta'''d_{001})\,\zeta. \end{aligned}$$

Therefore, eliminating $\xi$, $\eta$, $\zeta$, we have

$$\begin{vmatrix} \alpha + \alpha'''d_{100}, & \alpha' + \alpha'''d_{010}, & \alpha'' + \alpha'''d_{001}, & \xi' \\ \beta + \beta'''d_{100}, & \beta' + \beta'''d_{010}, & \beta'' + \beta'''d_{001}, & \eta' \\ \gamma + \gamma'''d_{100}, & \gamma' + \gamma'''d_{010}, & \gamma'' + \gamma'''d_{001}, & \zeta' \\ \delta + \delta'''d_{100}, & \delta' + \delta'''d_{010}, & \delta'' + \delta'''d_{001}, & \tau' \end{vmatrix} = 0 \quad \text{.....(46).}$$

* "The Transformation of Linear Partial Differential Operators by Extended Linear Continuous Groups." *Proc. Lond. Math. Soc.*, Vol. XXIX.

Write this equation in the form

$$\tau' = D_{100}\xi' + D_{010}\eta' + D_{001}\zeta',$$

and $D_{100}$, $D_{010}$, $D_{001}$ are immediately determined.

As previously, write

$$J = \begin{vmatrix} \alpha + \alpha'''d_{100}, & \alpha' + \alpha'''d_{010}, & \alpha'' + \alpha'''d_{001} \\ \beta + \beta'''d_{100}, & \beta' + \beta'''d_{010}, & \beta'' + \beta'''d_{001} \\ \gamma + \gamma'''d_{100}, & \gamma' + \gamma'''d_{010}, & \gamma'' + \gamma'''d_{001} \end{vmatrix} = \frac{\partial(X, Y, Z)}{\partial(x, y, z)}.$$

The increments $\xi$, $\eta$, $\zeta$, $\tau$ have only been temporarily assumed small for the purpose of finding $D_{100}$, $D_{010}$, $D_{001}$. Let them now be regarded as finite; then the determinant on the left of equation (46) will be the value of

$$J(\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta').$$

Multiply the first three columns by $-\xi$, $-\eta$, $-\zeta$, and add to the last. We then find, by means of equations (45),

$$J(\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta')$$

$$= (\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta) \begin{vmatrix} \alpha + \alpha'''d_{100}, & \alpha' + \alpha'''d_{010}, & \alpha'' + \alpha'''d_{001}, & \alpha''' \\ \beta + \beta'''d_{100}, & \beta' + \beta'''d_{010}, & \beta'' + \beta'''d_{001}, & \beta''' \\ \gamma + \gamma'''d_{100}, & \gamma' + \gamma'''d_{010}, & \gamma'' + \gamma'''d_{001}, & \gamma''' \\ \delta + \delta'''d_{100}, & \delta' + \delta'''d_{010}, & \delta'' + \delta'''d_{001}, & \delta''' \end{vmatrix}$$

Multiply the last column by $d_{100}$, $d_{010}$, $d_{001}$ and subtract from the first, second and third columns respectively. We then obtain the important equation

$$\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta' = \frac{M}{J}(\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta) \quad \text{.....(47).}$$

This theorem is the generalization of a statement by Prof. Elliott (*Proc. Lond. Math. Soc.*, Vol. XVIII., p. 147) made with reference to two independent variables when the linear transformation consists of a cyclical interchange.

§ 26. Now

$$V_X = (\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta')\left(\frac{\partial\tau'}{\partial\xi'} - D_{100}\right).$$

Therefore the transformed expression for $V_X$ is obtained from

$$\frac{1}{M}(\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta')\left(\frac{\partial\tau'}{\partial\xi'} - D_{100}\right)\left(\Delta''' - \Delta\frac{\partial\tau}{\partial\xi} - \Delta'\frac{\partial\tau}{\partial\eta} - \Delta''\frac{\partial\tau}{\partial\zeta}\right),$$

by expressing it in terms of $\xi$, $\eta$, $\zeta$. The first factor is transformed by equation (47) which gives

$$\tau' - D_{100}\xi' - D_{010}\eta' - D_{001}\zeta' = \frac{M}{J}(\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta).$$

Again, from this last equation

$$\frac{\partial\tau'}{\partial\xi'} - D_{100} = \frac{M}{J}\left\{\left(\frac{\partial\tau}{\partial\xi} - d_{100}\right)\frac{\partial\xi}{\partial\xi'} + \left(\frac{\partial\tau}{\partial\eta} - d_{010}\right)\frac{\partial\eta}{\partial\xi'} + \left(\frac{\partial\tau}{\partial\zeta} - d_{001}\right)\frac{\partial\zeta}{\partial\xi'}\right\},$$

where $\frac{\partial\xi}{\partial\xi'}$, $\frac{\partial\eta}{\partial\xi'}$, $\frac{\partial\zeta}{\partial\xi'}$ are to be obtained from equations (45). These equations give

$$1=\left(\alpha+\alpha'''\frac{\partial\tau}{\partial\xi}\right)\frac{\partial\xi}{\partial\xi'}+\left(\alpha'+\alpha'''\frac{\partial\tau}{\partial\eta}\right)\frac{\partial\eta}{\partial\xi'}+\left(\alpha''+\alpha'''\frac{\partial\tau}{\partial\zeta}\right)\frac{\partial\zeta}{\partial\xi'},$$

$$0=\left(\beta+\beta'''\frac{\partial\tau}{\partial\xi}\right)\frac{\partial\xi}{\partial\xi'}+\left(\beta'+\beta'''\frac{\partial\tau}{\partial\eta}\right)\frac{\partial\eta}{\partial\xi'}+\left(\beta''+\beta'''\frac{\partial\tau}{\partial\zeta}\right)\frac{\partial\zeta}{\partial\xi'},$$

$$0=\left(\gamma+\gamma'''\frac{\partial\tau}{\partial\xi}\right)\frac{\partial\xi}{\partial\xi'}+\left(\gamma'+\gamma'''\frac{\partial\tau}{\partial\eta}\right)\frac{\partial\eta}{\partial\xi'}+\left(\gamma''+\gamma'''\frac{\partial\tau}{\partial\zeta}\right)\frac{\partial\zeta}{\partial\xi'}.$$

Now let

$$I=\begin{vmatrix}\alpha+\alpha'''\frac{\partial\tau}{\partial\xi}, & \alpha'+\alpha'''\frac{\partial\tau}{\partial\eta}, & \alpha''+\alpha'''\frac{\partial\tau}{\partial\zeta}\\ \beta+\beta'''\frac{\partial\tau}{\partial\xi}, & \beta'+\beta'''\frac{\partial\tau}{\partial\eta}, & \beta''+\beta'''\frac{\partial\tau}{\partial\zeta}\\ \gamma+\gamma'''\frac{\partial\tau}{\partial\xi}, & \gamma'+\gamma'''\frac{\partial\tau}{\partial\eta}, & \gamma''+\gamma'''\frac{\partial\tau}{\partial\zeta}\end{vmatrix}$$

$$=\Delta'''-\Delta\frac{\partial\tau}{\partial\xi}-\Delta'\frac{\partial\tau}{\partial\eta}-\Delta''\frac{\partial\tau}{\partial\zeta};$$

and let $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$ denote the minors of the first row of $I$; therefore

$$I\frac{\partial\xi}{\partial\xi'}=\mathfrak{A},\quad I\frac{\partial\eta}{\partial\xi'}=\mathfrak{A}',\quad I\frac{\partial\zeta}{\partial\xi'}=\mathfrak{A}''.$$

Hence

$$I\left(\frac{\partial\tau'}{\partial\xi'}-D_{100}\right)=\frac{M}{J}\left\{\mathfrak{A}\left(\frac{\partial\tau}{\partial\xi}-d_{100}\right)+\mathfrak{A}'\left(\frac{\partial\tau}{\partial\eta}-d_{010}\right)+\mathfrak{A}''\left(\frac{\partial\tau}{\partial\zeta}-d_{001}\right)\right\}$$

$$=\frac{M}{J}\begin{vmatrix}\frac{\partial\tau}{\partial\xi}-d_{100}, & \frac{\partial\tau}{\partial\eta}-d_{010}, & \frac{\partial\tau}{\partial\zeta}-d_{001}\\ \beta+\beta'''\frac{\partial\tau}{\partial\xi}, & \beta'+\beta'''\frac{\partial\tau}{\partial\eta}, & \beta''+\beta'''\frac{\partial\tau}{\partial\zeta}\\ \gamma+\gamma'''\frac{\partial\tau}{\partial\xi}, & \gamma'+\gamma'''\frac{\partial\tau}{\partial\eta}, & \gamma''+\gamma'''\frac{\partial\tau}{\partial\zeta}\end{vmatrix}$$

$$=\frac{M}{J}\begin{vmatrix}\frac{\partial\tau}{\partial\xi}-d_{100}, & \frac{\partial\tau}{\partial\eta}-d_{010}, & \frac{\partial\tau}{\partial\zeta}-d_{001}\\ \beta+\beta'''d_{100}, & \beta'+\beta'''d_{010}, & \beta''+\beta'''d_{001}\\ \gamma+\gamma'''d_{100}, & \gamma'+\gamma'''d_{010}, & \gamma''+\gamma'''d_{001}\end{vmatrix}$$

$$=\frac{M}{J}\left\{A\left(\frac{\partial\tau}{\partial\xi}-d_{100}\right)+A'\left(\frac{\partial\tau}{\partial\eta}-d_{010}\right)+A''\left(\frac{\partial\tau}{\partial\zeta}-d_{001}\right)\right\}\quad\text{.....(48)},$$

where $A$, $A'$, $A''$,... are the minors of $J$.

The transformed expression for $V_X$ is therefore

$$\frac{M}{J^2}(\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta)\left\{A\left(\frac{\partial\tau}{\partial\xi} - d_{100}\right) + A'\left(\frac{\partial\tau}{\partial\eta} - d_{010}\right) + A''\left(\frac{\partial\tau}{\partial\zeta} - d_{001}\right)\right\}$$

$$= \frac{M}{J^2}(AV_x + A'V_y + A''V_z),$$

or, in the previous notation,

$$\frac{M}{J^2}(AV_1 + A'V_2 + A''V_3) \quad \text{.....(49).}$$

Similarly the transformations of $V_Y$ and $V_Z$ are

$$\frac{M}{J^2}(BV_1 + B'V_2 + B''V_3), \quad \frac{M}{J^2}(CV_1 + C'V_2 + C''V_3) \quad \text{.....(49a).}$$

Again
$$\omega_{XY} = \xi'\left(\frac{\partial\tau'}{\partial\eta'} - D_{010}\right),$$

and the transformation of $\omega_{XY}$ is therefore

$$\frac{1}{M}\xi'\left(\frac{\partial\tau'}{\partial\eta'} - D_{010}\right)\left(\Delta''' - \Delta\frac{\partial\tau}{\partial\xi} - \Delta'\frac{\partial\tau}{\partial\eta} - \Delta''\frac{\partial\tau}{\partial\zeta}\right)$$

$$= \frac{1}{J}(\alpha\xi + \alpha'\eta + \alpha''\zeta + \alpha'''\tau)\left\{B\left(\frac{\partial\tau}{\partial\xi} - d_{100}\right) + B'\left(\frac{\partial\tau}{\partial\eta} - d_{010}\right) + B''\left(\frac{\partial\tau}{\partial\zeta} - d_{001}\right)\right\},$$

as in equation (48),

$$= \frac{1}{J}\{(\alpha + \alpha'''d_{100})\xi + (\alpha' + \alpha'''d_{010})\eta + (\alpha'' + \alpha'''d_{001})\zeta + \alpha'''(\tau - d_{100}\xi - d_{010}\eta - d_{001}\zeta)\}$$

$$\left\{B\left(\frac{\partial\tau}{\partial\xi} - d_{100}\right) + B'\left(\frac{\partial\tau}{\partial\eta} - d_{010}\right) + B''\left(\frac{\partial\tau}{\partial\zeta} - d_{001}\right)\right\}.$$

Now let $G_1$, $G_2$, $G_3$ be operators defined by the equations

$$G_1 = \xi\left(\frac{\partial\tau}{\partial\xi} - d_{100}\right) = \Sigma p d_{pqr}\frac{\partial}{\partial d_{pqr}},$$

$$G_2 = \eta\left(\frac{\partial\tau}{\partial\eta} - d_{010}\right) = \Sigma q d_{pqr}\frac{\partial}{\partial d_{pqr}},$$

$$G_3 = \zeta\left(\frac{\partial\tau}{\partial\zeta} - d_{001}\right) = \Sigma r d_{pqr}\frac{\partial}{\partial d_{pqr}},$$

where in each case $p + q + r > 1$.

Then the transformation of $\omega_{XY}$ is

$$\frac{1}{J}\{(\alpha + \alpha'''d_{100})BG_1 + (\alpha' + \alpha'''d_{010})B'G_2 + (\alpha'' + \alpha'''d_{001})B''G_3\}$$

$$+ \frac{1}{J}\{(\alpha + \alpha'''d_{100})(B'\omega_{xy} + B''\omega_{xz}) + (\alpha' + \alpha'''d_{010})(B\omega_{yx} + B''\omega_{yz}) + (\alpha'' + \alpha'''d_{001})(B\omega_{zx} + B'\omega_{zy})\}$$

$$+ \frac{\alpha'''}{J}(BV_1 + B'V_2 + B''V_3) \quad \text{.....(50).}$$

If the function $F$ operated on is isobaric, and of weights $w_1$, $w_2$, $w_3$ in first, second and third suffixes, we have

$$G_1F = w_1F, \quad G_2F = w_2F, \quad G_3F = w_3F.$$

If further $w_1 = w_2 = w_3$, since

$$B(\alpha + \alpha'''d_{100}) + B'(\alpha' + \alpha'''d_{010}) + B''(\alpha'' + \alpha'''d_{001}) = 0,$$

we find for the transformation of $\omega_{XY}$, a linear function of the operators $V$ and $\omega$, viz.

$$\frac{1}{J}\{(\alpha+\alpha'''d_{100})(B'\omega_{xy}+B''\omega_{xz})+(\alpha'+\alpha'''d_{010})(B\omega_{yx}+B''\omega_{yz})+(\alpha''+\alpha'''d_{001})(B\omega_{zx}+B'\omega_{zy})\}$$

$$+\frac{\alpha'''}{J}(BV_1+B'V_2+B''V_3) \quad \text{.....(51).}$$

The transformations of the remaining operators $\omega_{YX}$, $\omega_{XZ}$, ... can be written down at once from this last expression.

§ 27. For example, suppose a cyclical interchange is made from $z$, $x$, $y$ with $z$ for dependent variable to $x$, $y$, $z$ with $x$ for dependent variable. Using the symbolical notation, let

$$V_1 = (\zeta - z_{10}\xi - z_{01}\eta)(\zeta_\xi - z_{10}); \qquad V_2 = (\zeta - z_{10}\xi - z_{01}\eta)(\zeta_\eta - z_{01});$$

$$\omega_{xy} = \xi(\zeta_\eta - z_{01}); \qquad \omega_{yx} = \eta(\zeta_\xi - z_{10});$$

$$V_1' = (\xi - x_{10}\eta - x_{01}\zeta)(\xi_\eta - x_{10}); \qquad V_2' = (\xi - x_{10}\eta - x_{01}\zeta)(\xi_\zeta - x_{01});$$

$$\omega_{yz}' = \eta(\xi_\zeta - x_{01}); \qquad \omega_{zy}' = \zeta(\xi_\eta - x_{10}).$$

The scheme of transformation is

$$X = 0 \,.\, x + 1 \,.\, y + 0 \,.\, z,$$
$$Y = 0 \,.\, x + 0 \,.\, y + 1 \,.\, z,$$
$$Z = 1 \,.\, x + 0 \,.\, y + 0 \,.\, z,$$

so that $M = 1$, and

$$J = \begin{vmatrix} 0 & 1 \\ z_{10} & z_{01} \end{vmatrix} = -z_{10}.$$

The transformations of the operators are therefore respectively, if for simplicity we suppose them to act on pure isobaric functions with equal partial weights,

$$V_1' = \frac{1}{z_{10}^2}(z_{01}V_1 - z_{10}V_2) \quad \text{.....(52a),}$$

$$V_2' = -\frac{1}{z_{10}^2}V_1 \quad \text{.....(52b),}$$

$$\omega_{yz}' = \frac{1}{z_{10}}\omega_{yx} \quad \text{.....(52c),}$$

$$\omega_{zy}' = -\frac{1}{z_{10}}(z_{01}V_1 - z_{10}V_2) - \frac{1}{z_{10}}\{z_{10}(-z_{10})\,\omega_{xy} + z_{01}\,.\,z_{01}\,.\,\omega_{yx}\}$$

$$= -\frac{1}{z_{10}}(z_{01}V_1 - z_{10}V_2) + \frac{1}{z_{10}}(z_{10}^2\omega_{xy} - z_{01}^2\omega_{yx}) \quad \text{.....(52d).}$$

§ 28. As an application of the last results we may employ them to prove two formulae which are fundamental in Prof. Elliott's paper "On the Linear Partial Differential Equations satisfied by Ternary Reciprocants," *Proc. Lond. Math. Soc.*, Vol. XVIII.

Let $F(z_{pq}, \ldots)$ be a pure function of the differential coefficients, and let $\mu$ be any number. It is required to evaluate

$$\frac{\partial}{\partial x_{01}}\left(\frac{F}{z_{10}{}^{\mu}}\right) \text{ and } \frac{\partial}{\partial x_{10}}\left(\frac{F}{z_{10}{}^{\mu}}\right),$$

where in the differentiation with respect to $x_{01}$ it is assumed that $x_{10}, x_{20}, x_{11}, \ldots$ are regarded as constant, and, in the differentiation with respect to $x_{10}, x_{01}, x_{20}, x_{11}, \ldots$ are constant. We shall, for simplicity, assume $F$ to be homogeneous of degree $i$ and isobaric with equal partial weights $w$.

Now the change from $x, yz$ to $z, xy$ is the second cyclical interchange from $x, yz$; therefore by making suitable interchanges of letters in equation (43*b*) we have

$$F(z_{pq}, \ldots) = (-1)^i \frac{1}{x_{01}{}^{i+w}} e^{-\frac{V_2'}{x_{01}}} e^{-\frac{x_{10}}{x_{01}}\omega'_{yz}} F(x_{qp}, \ldots),$$

where $V_2'$ and $\omega_{yz}'$ are the same operators as in the preceding section. It may be remarked that $V_2'$ and $\omega_{yz}'$ are commutative by § 21. Therefore since $z_{10} = \frac{1}{x_{01}}$,

$$\frac{\partial}{\partial x_{01}}\left(\frac{F}{z_{10}{}^{\mu}}\right) = \left\{\frac{\mu - i - w}{x_{01}} + \frac{1}{x_{01}{}^2} V_2' + \frac{x_{10}}{x_{01}{}^2}\omega_{yz}'\right\} . (-1)^i x_{01}{}^{\mu-i-w} e^{-\frac{V_2'}{x_{01}}} e^{-\frac{x_{10}}{x_{01}}\omega_{yz}'} F(x_{qp}, \ldots)$$

$$= \left\{\frac{\mu - i - w}{z_{10}{}^{\mu-1}} - \frac{1}{z_{10}{}^{\mu}} V_1 - \frac{z_{01}}{z_{10}{}^{\mu}}\omega_{yz}\right\} F(z_{pq}, \ldots),$$

by (52*b*) and (52*c*).

Similarly

$$\frac{\partial}{\partial x_{10}}\left(\frac{F}{z_{10}{}^{\mu}}\right) = \frac{1}{z_{10}{}^{\mu}}\left(-\frac{1}{x_{01}}\right)\omega_{yz}' . (-1)^i \frac{1}{x_{01}{}^{i+w}} e^{-\frac{V_2'}{x_{01}}} e^{-\frac{x_{10}}{x_{01}}\omega_{yz}'} F(x_{qp}, \ldots)$$

$$= -\frac{1}{z_{10}{}^{\mu}}\omega_{yz} F(z_{pq}, \ldots),$$

by (52*c*).

These are Prof. Elliott's formulae.

§ 29. The theory of cyclicants is a generalization of the theory of ordinary reciprocants; in the case when there are two independent variables it plays a part which has the same reference to the theory of surfaces that the ordinary reciprocant has to the theory of plane curves. But the ordinary reciprocant may be looked at from another point of view. Regarding $y$ as a function of $x$, let us suppose $\xi$, $\eta$ to

be corresponding increments of $x$ and $y$; then writing $a_n$ for $\frac{1}{n!}\frac{d^n y}{dx^n}$ and $A_n$ for $\frac{1}{n!}\frac{d^n x}{dy^n}$, we have

$$\eta = a_1\xi + a_2\xi^2 + \dots,$$

$$\xi = A_1\eta + A_2\eta^2 + \dots.$$

The second series is that obtained by reversion of the first, and a reciprocant may be looked upon as a function of the coefficients of a series which is unaltered in value, save for a factor involving $a_1$, when the series is reversed. From this point of view the generalized reciprocant may be defined in the following manner. Using the notation of § 16 let $u$, $v$, $w$ be functions of $x$, $y$, $z$, and let $a_{pqr}$ denote $\frac{1}{p!\,q!\,r!}\frac{\partial^{p+q+r}u}{\partial x^p \partial y^q \partial z^r}, \dots$ and let $A_{pqr}$ denote $\frac{1}{p!\,q!\,r!}\frac{\partial^{p+q+r}x}{\partial u^p \partial v^q \partial w^r}$. Then $F(a_{pqr}, b_{pqr}, c_{pqr}, \dots)$ will be a reciprocant if

$$F(A_{pqr}, B_{pqr}, C_{pqr}, \dots) = \mu F(a_{pqr}, b_{pqr}, c_{pqr}, \dots),$$

where $\mu$ is a function of first differential coefficients only. The function will be called an $n$-ary reciprocant if there are $n$ independent variables involved, and $F$ will be a pure function if it is free from first differential coefficients. This kind of reciprocant may also be regarded as a function of the coefficients of series which is unaltered, save for a factor, when the series are reversed and the coefficients of the reversed series are substituted for those of the original series.

Sufficient conditions to ensure the permanence of such functions, when pure, are easily obtained from the results of § 16.

Reciprocants of the kind here considered have been discussed by Prof. Elliott* for the case of two independent variables. The conditions here obtained for $n$ variables agree with those obtained by Prof. Elliott, who however does not examine into the question of the independence of his conditions.

Suppose $F(a_{lmn}, b_{lmn}, c_{lmn}, \dots)$ to be a homogeneous function of degree $i$ and isobaric with equal partial weights $w$. Then

$$F(A_{lmn}, B_{lmn}, C_{lmn}, \dots) = \frac{(-1)^i}{J^{i+w}} e^{-[U,V,W]} e^{\Omega_1} e^{\Omega_2} e^{\Omega_3}.$$

$$F(Aa_{lmn} + Bb_{lmn} + Cc_{lmn},\ A'a_{lmn} + B'b_{lmn} + C'c_{lmn},\ A''a_{lmn} + B''b_{lmn} + C''c_{lmn}, \dots) \quad \dots(53).$$

The function will therefore be permanent in form if it is an invariant of the system of emanants

$$\left.\begin{array}{ll} a_{200}\xi^2 + a_{020}\eta^2 + a_{002}\zeta^2 + a_{110}\xi\eta + \dots, & a_{300}\xi^3 + \dots,\ \dots \\ b_{200}\xi^2 + \dots \qquad\qquad\qquad\qquad , & b_{300}\xi^3 + \dots,\ \dots \\ c_{200}\xi^2 + \dots \qquad\qquad\qquad\qquad , & c_{300}\xi^3 + \dots,\ \dots \end{array}\right\} \dots\dots\dots\dots\dots\dots(54),$$

* "On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables." *Proc. Lond. Math. Soc.*, Vol. XXII. pp. 79—104.

which remains an invariant when $\lambda u + \mu v + \nu w$, $\lambda' u + \mu' v + \nu' w$, $\lambda'' u + \mu'' v + \nu'' w$ are substituted for $u$, $v$, $w$, and which is further annihilated by the various operators $[U, x]$, $[V, x]$, $[W, x]$, $[U, y]$, ... . The operators as defined in § 14 contain terms with differential operators involving $t$; such terms will of course be omitted here.

It is obvious that functions which satisfy the conditions just laid down will be unchanged or at most changed only in sign when $u$, $v$, $w$ are interchanged; and that they will be unchanged or changed only in sign if first, second and third suffixes are interchanged. Such functions therefore, if homogeneous, will be of equal partial degrees in $a_{pqr}, \ldots, b_{pqr}, \ldots, c_{pqr}, \ldots$ .

When linear functions $\lambda u + \mu v + \nu w, \ldots$ are substituted for $u$, $v$, $w$ in a combinant the function is multiplied by the $i$th power of

$$\begin{vmatrix} \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \\ \lambda'', & \mu'', & \nu'' \end{vmatrix},$$

where $i$ is equal to any one of the equal partial degrees of the combinant.

In the case of the reciprocants here considered the determinant is

$$\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix},$$

which is equal to $J^2$; and the determinant is equal to $J^{n-1}$ when there are $n$ independent variables.

Reference to equation (32) then shows that the factor for a reciprocant of equal partial degrees $i$ and equal partial weights $w$ is

$$\frac{(-1)^{ni} J^{(n-1)i}}{J^{ni+w}} = \frac{(-1)^{ni}}{J^{i+w}},$$

so that

$$F(A_{pqr}, B_{pqr}, C_{pqr}, \ldots) = \frac{(-1)^{ni}}{J^{i+w}} F(a_{pqr}, b_{pqr}, c_{pqr}, \ldots).$$

§ 30. One example of such reciprocants is easily seen to be the eliminant of the quadratic emanants just written down. For this is an invariant of the required type, and since it involves no differential coefficients $a_{pqr}$ for which $p+q+r > 2$, it is obviously annihilated by $[U, x]$, $[V, x]$, ... .

This example for the case of two independent variables is given by Prof. Elliott. The eliminant in this case is

$$(a_{20}b_{11} - b_{20}a_{11})(a_{11}b_{02} - b_{11}a_{02}) - (a_{20}b_{02} - b_{20}a_{02})^2.$$

The partial degree $i=2$, and the partial weight $w=4$; therefore by the last result of § 29 this expression is equal to

$$J^6\{(A_{20}B_{11}-B_{20}A_{11})(A_{11}B_{02}-B_{11}A_{02})-(A_{20}B_{02}-B_{20}A_{02})^2\},$$

where
$$J=ab'-a'b.$$

The invariant character of the function just considered corresponds to a simple theorem in the theory of the reversion of series. Let

$$\upsilon=a_{10}\xi+a_{01}\eta+a_{20}\xi^2+a_{11}\xi\eta+a_{02}\eta^2+\dots,$$

$$\nu=b_{10}\xi+b_{01}\eta+b_{20}\xi^2+b_{11}\xi\eta+b_{02}\eta^2+\dots,$$

and suppose that from these equations are deduced the series

$$\xi=A_{10}\upsilon+A_{11}\nu+A_{20}\upsilon^2+A_{11}\upsilon\nu+A_{02}\nu^2+\dots,$$

$$\eta=B_{10}\upsilon+B_{11}\nu+B_{20}\upsilon^2+B_{11}\upsilon\nu+B_{02}\nu^2+\dots.$$

The theorem then states that, if the quadratic terms in the original series have a common factor linear in $\xi$, $\eta$, the quadratic terms of the series obtained by reversion will have a quadratic factor linear in $\upsilon$, $\nu$.

The theorem is easily proved independently. The property referred to is one unaltered by a linear transformation, and therefore we may take $\xi$ for the common factor. The method of successive approximation then shows at once that the quadratic terms in the last two series must have a common factor.

§ 31. The conditions for pure reciprocants laid down in § 27 although sufficient are not independent. This statement can be proved by forming the alternants of various operators. If we assume $F$ to be an invariant of the system of emanants (54) which remains unaltered save for a factor when $\lambda u+\mu v+\nu w$, $\lambda' u+\mu' v+\nu' w$, $\lambda'' u+\mu'' v+\nu'' w$ are substituted for $u$, $v$, $w$, then it can be shown that annihilation by one of the operators $[U, x]$ will ensure annihilation by all the others. In fact since $F$ is invariant when $\lambda u+\mu v+\nu w$ is substituted for $u$, therefore $F$ must be annihilated by the operators which in the usual symbolical notation will be denoted by $[V_1]$ and $[W_1]$, so that

$$[V_1]=[b_{200}\xi_1^2+\dots]=b_{200}\frac{\partial}{\partial a_{200}}+\dots,$$

$$[W_1]=[c_{200}\xi_1^2+\dots]=c_{200}\frac{\partial}{\partial a_{200}}+\dots.$$

Similarly $F$ must be annihilated by $[U_2]$, $[W_2]$, $[U_3]$, $[V_3]$.

Now, in the present case, we have

$$[U, x]=[U_1U_{\xi_1}]+[U_2V_{\xi_2}]+[U_3W_{\xi_3}];$$

therefore

$$[U, x][V_1]-[V_1][U, x]=[U_2V_{\xi_2}]*[V_1]-[V_1]*[U_1U_{\xi_1}]-[V_1]*[U_2V_{\xi_2}]-[V_1]*[U_3W_{\xi_3}].$$

And $$[V_1]\,U_1 = V_1, \quad [V_1]\,U_2 = V_2, \quad [V_1]\,U_{\xi_1} = \frac{\partial}{\partial \xi_1}[V_1]\,U_1 = \frac{\partial V_1}{\partial \xi_1}.$$

Therefore

$$\begin{aligned}[U, x][V_1] - [V_1][U, x] &= [U_1 V_{\xi_1}] - [U_{\xi_1} V_1 + U_1 V_{\xi_1}] - [V_2 V_{\xi_2}] - [V_3 W_{\xi_3}] \\ &= -[V_1 U_{\xi_1}] - [V_2 V_{\xi_2}] - [V_3 W_{\xi_3}] \\ &= -[V, x] \quad \text{.....(55}a\text{).}\end{aligned}$$

Similarly

$$[U, x][W_1] - [W_1][U, x] = -[W, x] \quad \text{.....(55}b\text{);}$$

and other equations can be written down with $y$, $z$ in place of $x$, and with $U$, $V$, $W$ interchanged.

Again $$\Omega_{xy} = [\xi_1 U_{\eta_1}] + [\xi_2 V_{\eta_2}] + [\xi_3 W_{\eta_3}];$$

therefore

$$\begin{aligned}\Omega_{xy}[U, x] - [U, x]\,\Omega_{xy} &= [\xi_1 U_{\eta_1}] * [U_1 U_{\xi_1}] + [\xi_1 U_{\eta_1}] * [U_2 V_{\xi_2}] + [\xi_1 U_{\eta_1}] * [U_3 W_{\xi_3}] \\ &\quad + [\xi_2 V_{\eta_2}] * [V_2 V_{\xi_2}] + [\xi_3 W_{\eta_3}] * [U_3 W_{\xi_3}] \\ &\quad - [U_1 U_{\xi_1}] * [\xi_1 U_{\eta_1}] - [U_2 V_{\xi_2}] * [\xi_2 V_{\eta_2}] - [U_3 W_{\xi_3}] * [\xi_3 W_{\eta_3}] \\ &= \left[\xi_1 U_{\eta_1} U_{\xi_1} + U_1 \frac{\partial}{\partial \xi_1}(\xi_1 U_{\eta_1})\right] + [\xi_2 U_{\eta_2} V_{\xi_2}] + [\xi_3 U_{\eta_3} W_{\xi_3}] + \left[U_2 \frac{\partial}{\partial \xi_2}(\xi_2 V_{\eta_2})\right] \\ &\quad + \left[U_3 \frac{\partial}{\partial \xi_3}(\xi_3 W_{\eta_3})\right] - \left[\xi_1 \frac{\partial}{\partial \eta_1}(U_1 U_{\xi_1})\right] - \left[\xi_2 \frac{\partial}{\partial \eta_2}(U_2 V_{\xi_2})\right] - \left[\xi_3 \frac{\partial}{\partial \eta_3}(U_3 W_{\xi_3})\right] \\ &= [U_1 U_{\eta_1}] + [U_2 V_{\eta_2}] + [U_3 W_{\eta_3}] \\ &= [U, y] \quad \text{.....(55}c\text{).}\end{aligned}$$

Similarly

$$\Omega_{xz}[U, x] - [U, x]\,\Omega_{xz} = [U, z] \quad \text{.....(55}d\text{);}$$

and other equations can also be obtained with $V$, $W$ in place of $U$, or with $x$, $y$, $z$ interchanged.

Equations (55$a$), (55$b$) show that any function annihilated by $[U, x]$, $[V_1]$ and $[W_1]$ will also be annihilated by $[V, x]$ and $[W, x]$; and then equations (55$c$) and (55$d$), with similar equations in which $V$, $W$ are written for $U$, show that the function will also be annihilated by $[U, y]$, $[V, y]$, $[W, y]$, $[U, z]$, $[V, z]$, $[W, z]$.

Hence defining a combinant of the emanants (54) as an invariant which remains invariant when $u$, $v$, $w$ are replaced by linear functions of $u$, $v$, $w$, we see that any combinant of the emanants which is annihilated by any one of the operators $[U, x]$, $[U, y]$, ... will be annihilated by all the others and will therefore be a reciprocant in the sense defined in § 29.

# IX. *On Divergent Hypergeometric Series.* By Prof. W. McF. Orr, M.A., Royal College of Science, Dublin.

*Addition* *. [*Received* 3 April 1899.]

13. We have obtained the complete solution of equation (3) in divergent series only in the case in which $m=n+1$. It has been shown by Stokes (*Camb. Phil. Soc. Proc.* Vol. VI.) that in any case in which $m<n+1$, as $x$ increases indefinitely, remaining real and positive, the ratio of

$$\frac{\Pi(\alpha_1-1)\,\Pi(\alpha_2-1)\ldots\Pi(\alpha_n-1)}{\Pi(\rho_1-1)\,\Pi(\rho_2-1)\ldots\Pi(\rho_n-1)}F(\alpha_1,\alpha_2,\ldots\alpha_m;\,\rho_1,\ldots\rho_n;\,x)$$

to

$$(n+1-m)^{-\frac{1}{2}}(2\pi y)^{\frac{m-n}{2}}y^{n-m+\Sigma\alpha-\Sigma\rho}\,e^{(n+1-m)y}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(60),$$

where $y^{n+1-m}=x$, has unity for its limit. The form of this expression, which admits for a complex $x$ of $n+1-m$ independent values, suggests that $n+1-m$ (the missing number) further independent solutions may be obtained each as the product of one of the values of (60) by a divergent series proceeding in descending powers of $y$. The form of this series, even when no attention is paid to its arithmetical significance, is somewhat complicated even for the case in which $m=n$ and its complexity increases with every increase in the value of $n+1-m$. We will therefore content ourselves with establishing the forms towards which as $x$ increases indefinitely the equations connecting the convergent and divergent functions tend. It may be convenient to use Lord Kelvin's symbol for approximate equality, viz. $\doteqdot$, to denote that under certain circumstances, obvious from the context, the ratio of the two expressions it connects may be made as nearly as desired equal to unity.

It may be readily proved by induction that when $x$ is great

$$\frac{\Pi(-\rho_1)\,\Pi(-\rho_2)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\,\Pi(-\alpha_2)\ldots\Pi(-\alpha_n)}F(\alpha_1,\alpha_2,\ldots\alpha_n;\,\rho_1,\rho_2,\ldots\rho_n;\,-x)$$

$$+\frac{\Pi(\rho_1-2)\,\Pi(\rho_1-\rho_2-1)\ldots\Pi(\rho_1-\rho_n-1)}{\Pi(\rho_1-\alpha_1-1)\,\Pi(\rho_1-\alpha_2-1)\ldots\Pi(\rho_1-\alpha_n-1)}x^{1-\rho_1}F(\alpha_1-\rho_1+1,\ldots\alpha_n-\rho_1+1;$$

$$2-\rho_1,\rho_2-\rho_1+1,\ldots\rho_n-\rho_1+1;\,-x)$$

$+(n-1)$ other terms analogous to the last

$$=e^{-x}x^{\Sigma\alpha-\Sigma\rho}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(61),$$

* See *Trans. Camb. Phil. Soc.*, Vol. XVII. Part III. p. 171.

the argument of every power $x^m$ lying between $-m\pi$ and $+m\pi$. (As limiting forms of this equation are required, it may be well to examine its genesis more fully.) In establishing this result we may follow the procedure of Art. 8; we consider

$$x^{1-\rho}\int (v-x)^{\rho-\alpha-1}\, v^{\alpha-1}\, \phi(-v)\, dv \quad \text{.....(62)},$$

where $\phi(-v)$ is a solution of the equation satisfied by $F(\alpha_1, \dots \alpha_n; \rho_1, \dots \rho_n; -v)$ and the path of integration is the same as in Fig. 3.

Equations (38) are now to be replaced by

$$\left.\begin{aligned} C\,(e^{2\pi i(\alpha-\rho)} - e^{2\pi i\alpha}) &= \frac{\Pi(-\rho_1)\dots\Pi(-\rho_n)}{\Pi(-\alpha_1)\dots\Pi(-\alpha_n)} \\ C_r\,(e^{2\pi i(\alpha-\rho)} - e^{2\pi i(\alpha-\rho_r)}) &= \frac{\Pi(\rho_r-2)\,\Pi(\rho_r-\rho_1-1)\dots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha_1-1)\dots\Pi(\rho_r-\alpha_n-1)} \end{aligned}\right\} \quad \text{.....(63)};$$

hence if we assume equation (61) to hold for the function of the $(n+1)$th order, the integral remains finite for each of the four paths considered without any restriction on the values of the $\alpha$'s and $\rho$'s. The expression which replaces (39) may be written in the form

$$\int_{\infty}^{(x)} x^{1-\rho}(v-x)^{\rho-\alpha-1}\, v^{\alpha-1}\, \psi(v) \quad \text{.....(64)},$$

where as $v$ increases indefinitely $\psi(v) \doteqdot e^{-v}\, v^{\sum_1^n(\alpha_r-\rho_r)}$ and initially the argument of $\psi(v)$ is zero. It may be shown as in Art. 5 that when $x$ is increased indefinitely this tends to equality with

$$(e^{2\pi i(\rho-\alpha)}-1)\,\Pi(\rho-\alpha-1)\, e^{-x}\, x^{\alpha-\rho+\sum_1^n(\alpha_r-\rho_r)} \quad \text{.....(65)}.$$

The expression which now replaces (41) may be shown to be equal to

$$-(e^{2\pi i(\rho-\alpha)}-1)\frac{\Pi(\rho-2)\,\Pi(\rho-\rho_1-1)\dots\Pi(\rho-\rho_n-1)}{\Pi(\rho-\alpha_1-1)\,\Pi(\rho-\alpha_2-1)\dots\Pi(\rho-\alpha_n-1)}\, x^{1-\rho}\, F(\alpha-\rho+1,$$
$$\alpha_1-\rho+1, \dots \alpha_n-\rho+1;\, 2-\rho, \rho_1-\rho+1, \dots \rho_n-\rho+1;\, -x) \quad \text{.....(66)}.$$

The value of the whole integral (62) is now, instead of (46),

$$(e^{2\pi i(\rho-\alpha)}-1)\,\Pi(\rho-\alpha-1)\left[\frac{\Pi(-\rho)\,\Pi(-\rho_1)\dots\Pi(-\rho_n)}{\Pi(-\alpha)\,\Pi(-\alpha_1)\dots\Pi(-\alpha_n)}\, F(\alpha, \dots \alpha_n; \rho, \dots \rho_n; -x)\right.$$
$$+\sum_{r=1}^{r=n}\frac{\Pi(\rho_r-2)\,\Pi(\rho_r-\rho_1-1)\dots\Pi(\rho_r-\rho_n-1)}{\Pi(\rho_r-\alpha-1)\,\Pi(\rho_r-\alpha_1-1)\dots\Pi(\rho_r-\alpha_n-1)}\, x^{1-\rho_r}\, F(\alpha-\rho_r+1, \dots$$
$$\left.\alpha_n-\rho_r+1;\, 2-\rho_r, \rho-\rho_r+1, \dots \rho_n-\rho_r+1;\, -x)\right] \quad \text{.....(67)}.$$

Equating (67) to the sum of (65) and (66) and dividing by $(e^{2\pi i(\rho-\alpha)}-1)\,\Pi(\rho-\alpha-1)$ an equation is obtained of the same type as (61) but with an additional $\alpha$ and an additional $\rho$. This equation, omitted through considerations of space, we will number (68).

Equation (61) holds even if the limits for the argument of $x$ be extended to $-3\pi/2$

and $+3\pi/2$, for the difference between the two values of the left-hand member which are thus stated to be approximately equal to the same multiple of the right-hand member may be shown to be a linear function of the $n$ solutions which when $x$ is great are respectively of the orders $x^{-\alpha_1}, \dots x^{-\alpha_n}$, and therefore when the real part of $x$ is negative is very small compared with the right-hand member.

We now desire to find the limiting form of (68) when we write $\alpha = -h$, $x = y/h$, then make $h$ increase indefinitely and finally suppose the modulus of $y$ to be large. Multiplying the equation by $\Pi(h)$ the limiting form of the left-hand member, whatever be the modulus of $y$, is

$$\frac{\Pi(-\rho)\,\Pi(-\rho_1)\dots\Pi(-\rho_n)}{\Pi(-\alpha_1)\,\Pi(-\alpha_2)\dots\Pi(-\alpha_n)} F(\alpha_1, \dots \alpha_n;\ \rho,\ \rho_1, \dots \rho_n;\ +y)$$

$$+\frac{\Pi(\rho-2)\,\Pi(\rho-\rho_1-1)\dots\Pi(\rho-\rho_n-1)}{\Pi(\rho-\alpha_1-1)\dots\Pi(\rho-\alpha_n-1)} y^{1-\rho} F(\alpha_1-\rho+1, \dots \alpha_n-\rho+1;\ 2-\rho,$$

$$\rho_1-\rho+1, \dots \rho_n-\rho+1;\ +y)$$

$+n$ other terms analogous to the last ..... (69);

while the right-hand member, *i.e.* a certain multiple of (64), for any value of $y$ becomes

$$\frac{\Pi(h)}{\Pi(\rho+h-1)} \int_{y/h}^{\infty} (y/h)^{1-\rho} (v-y/h)^{\rho+h-1} v^{-h-1} \psi(v),$$

or as $h$ increases indefinitely

$$y^{1-\rho} \int_{\epsilon y}^{\infty} v^{\rho-2} e^{-y/v} \psi(v) \quad \dots\dots\dots\dots (70),$$

where $\psi(v)$ when $v$ is small is of the order of a power of $v$, and when $v$ is great is approximately equal to $e^{-v} v_1^{\frac{n}{2}(\alpha_r-\rho_r)}$, provided the argument of $v$ lies between $-3\pi/2$ and $+3\pi/2$, and $\epsilon$ is an indefinitely small positive quantity. It should be noted that the limitations placed on the argument of $x$ in the integrals which have been expressed by divergent series were only imposed in order to make those series arithmetically intelligible in the sense of equation (28), but that while the integral forms are retained no such limitations are necessary. We may accordingly suppose that in (70) the limits of the argument of $y$ are still further extended to $-2\pi$ and $+2\pi$; for in evaluating

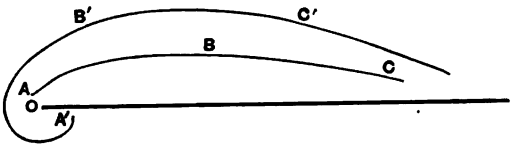



Fig. 6.

the integral when the argument of $y$ lies between $-3\pi/2$ and $-2\pi$ we may change the lower limit to a point whose argument is $-3\pi/2$ without altering its value, and so have all along the path of integration $\psi(v) = e^{-v} v^{\frac{n}{2}(\alpha_r-\rho_r)}$. As regards the path of

$v$ a consideration of (37) of which it is a limiting form, and of Fig. 3, shows that if the argument of $y$ lies between 0 and $2\pi$ the path must be such as $ABC$, or $A'B'C'$, (Fig. 6), while if the argument lies between 0 and $-2\pi$ the path must be such as the image of this with respect to the axis of real quantities.

We can take $y$ so great that $\psi(v)$ is as nearly equal as we please to $e^{-v}v_1^{\frac{n}{2}(a_r-\rho_r)}$ for values of $v$ for which $e^{-y/v}$ is as small as we please, and accordingly so that the value of (70) is as nearly as we please equal to that obtained by replacing $\psi(v)$ by this approximation. We would then have to consider an integral of the type

$$\int_{\epsilon y}^{\infty} e^{-v-y/v}\, v^p\, dv \quad \text{.....} \quad (71).$$

This is a particular case of another with which we will have to deal, viz.:—

$$\int_{\epsilon y}^{\infty} e^{-sv-y^s v^{-s}} \,.\, v^p dv \quad \text{.....} \quad (72),$$

$s$ being a positive integer, and $\epsilon$ an indefinitely small positive quantity, the argument of $v$ at the infinite limit being zero, and the argument of $y$ lying between $-(s+1)\pi/s$ and $+(s+1)\pi/s$, the path of integration thus being permitted to make round the origin a number of revolutions determined by the initial argument. See Fig. 7, in

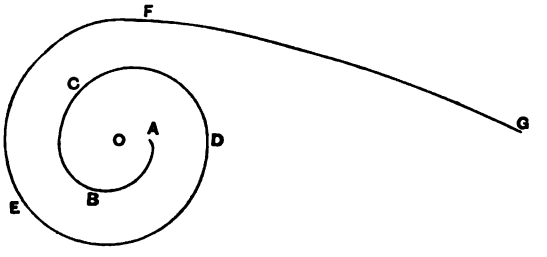



Fig. 7.

which $ABCDEFG$ represents a case in which the argument of $y$ is positive, as we will at first suppose.

The value of $e^{-sv-y^s v^{-s}}$ is stationary for values of $v$ given by the equation

$$1 - y^s v^{-s-1} = 0 \quad \text{.....} \quad (73);$$

let $v_1$ be that root whose argument is $s/(s+1)$ times the argument of $y$, and thus lies between 0 and $+\pi$. It may be noted that if the point corresponding to any other solution of (73) lies in the region traversed by the path of integration in (72), the real part of $v$ at any such point is very much greater algebraically than the real part of $v_1$, and therefore the modulus of $e^{-sv-y^s v^{-s}}$ very much less than that of $e^{-sv_1-y^s v_1^{-s}}$. We will now suppose the path of integration to pass through $v_1$, and consider separately the two portions of the path from $v_1$ to $\infty$ and from $\epsilon y$ to $v_1$. Considering the former, let part of it be a straight line starting from $v_1$ in a direction whose argument is

half that of $v_1$, a direction which makes an acute angle with the positive part of the axis of real quantities and an obtuse angle with the line joining $v_1$ to the origin.

Expanding $v^{-s}$ in powers of $v - v_1$ we have for points on this line

$$sv + y^s v^{-s} = (s+1)\, v_1 + s\,(s+1)\,(v - v_1)^2/2v_1 + R,$$

where mod. $R < s\,(s+1)\,(s+2)\,(v - v_1)^3/6v_1^2$ (see Art. 2). We can thus take $y$ so great that along this line $R$ is less than any assigned quantity for a range such that throughout it the ratio of $v$ to $v_1$ is as nearly as we please equal to unity, and at the end of it the term $s\,(s+1)\,(v - v_1)^2/2v_1$ is greater than any assigned quantity. If $v_2$ denote the value of $v$ at the end of the range this may be done by increasing $v_1$ and making $v_2 - v_1$ vary as $v_1^{7/12}$ (say). Along such a range then $v^p\, e^{-sv - y^s v^{-s}}$ is as nearly as we please equal to

$$v_1^p\, e^{-(s+1)\, v_1}\, e^{-s\,(s+1)\,(v - v_1)^2/2v_1},$$

while at the end of it the final factor of the last expression is less than any assigned quantity, since $(v - v_1)^2/2v_1$ is real and positive.

The portion contributed to the integral (72) by this range is thus as nearly as we please equal to

$$v_1^p\, e^{-(s+1)\, v_1} \int_0^\infty e^{-s\,(s+1)\,(v - v_1)^2/2v_1} \quad \text{.....} \quad (74),$$

or

$$\left(\frac{\pi v_1}{2s\,(s+1)}\right)^{\frac{1}{2}} v_1^p\, e^{-(s+1)\, v_1} \quad \text{.....} \quad (75),$$

wherein the argument of $v_1^{p+1/2}$ is $p + 1/2$ times the argument of $v_1$.

We next proceed to show that the portion contributed to the integral by the path from $v_2$ to $\infty$ can at the same time be made less than any assigned quantity. Consider the expression

$$sv + y^s v^{-s} - \{sv_2 + y^s v_2^{-s} + s\,(v - v_2)\,(1 - c)\} \quad \text{.....} \quad (76),$$

where $c$ is real, positive, and less than unity. This expression vanishes when $v = v_2$, and its real part is infinitely great and positive when, and only when, $v$ is infinite and has its real part positive. It is therefore evident that a curve can be drawn from $v_2$ to $\infty$, in the negative direction round the origin, such that along it the real part of (76) increases continually. Everywhere along such a curve we would have

$$\text{mod. } e^{-sv - y^s v^{-s}} < \text{mod. } e^{-sv_2 - y^s v_2^{-s}}\, e^{-s\,(v - v_2)\,(1 - c)}.$$

The part contributed to the integral by this curve would be increased if each term were replaced by its modulus, but if the argument of $v_1$ is $\pi - \alpha$ this would replace

$$\int_{v_2}^\infty e^{-s\,(v - v_2)\,(1 - c)}\, v^p dv$$

by something certainly less than

$$\Pi\,(p) \left\{ s\,(1 - c) \sin \frac{\alpha}{2} \right\}^{-p-1}$$

which is finite while the factor $e^{-sv_2 - y_s v_2^{-s}}$ is less than any assigned quantity, therefore this part is less than any assigned quantity.

We next consider the portion of the integral contributed by the path from $ey$ to $v_1$. By means of the substitutions

$$y^s v^{-s} = us^{-1},\ s^{\frac{s+1}{s}}\, y = y_1^{\frac{1}{s}}$$

this portion can be made to depend on an integral similar to that just discussed, except that $s$ and $1/s$ are interchanged, and can be shown to be also equal to (75) when $y$ is very great.

Thus (72) tends to equality with twice (75); and a similar result is true when the argument of $y$ is negative.

We now write in (75)

$$s = 1,\ p = \rho - 2 + \sum_1^n (\alpha_r - \rho_r),$$

and accordingly when $y$ is increased indefinitely (70) tends to equality with

$$\pi^{\frac{1}{2}} y^{\frac{1}{2}(\Sigma\alpha - \Sigma\rho + \frac{1}{2})}\, e^{-2y^{\frac{1}{2}}},$$

$\Sigma\rho$ now including $\rho$.

This result refers to the function of the $(n+2)$th order. If we now reduce the order by unity, omitting $\alpha_n$ and $\rho$, and change $y$ into $x$, we have the equation

$$\frac{\Pi(-\rho_1)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\ldots\Pi(-\alpha_{n-1})} F(\alpha_1, \ldots \alpha_{n-1};\ \rho_1, \ldots \rho_n;\ +x)$$

$$+ \frac{\Pi(\rho_1 - 2)\,\Pi(\rho_1 - \rho_2 - 1)\ldots\Pi(\rho_1 - \rho_n - 1)}{\Pi(\rho_1 - \alpha_1 - 1)\ldots\Pi(\rho_1 - \alpha_{n-1} - 1)} x^{1-\rho_1} F(\alpha_1 - \rho_1 + 1, \ldots \alpha_{n-1} - \rho_1 + 1;$$

$$2 - \rho_1,\ \rho_2 - \rho_1 + 1, \ldots \rho_n - \rho_1 + 1;\ +x)$$

$$+ (n-1) \text{ other terms analogous to the last}$$

$$= \pi^{\frac{1}{2}} x^{\frac{1}{2}(\Sigma\alpha - \Sigma\rho + \frac{1}{2})}\, e^{-2x^{\frac{1}{2}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (77).$$

As this relation holds while the argument of $x$ ranges from $-2\pi$ to $+2\pi$, it is equivalent to two independent relations among the functions considered.

It also holds even when the limits of the argument are extended to $-3\pi$ and $+3\pi$, for the difference between the two values of the left-hand member which are thus stated to be approximately equal to the same multiple of the right-hand member may be shown to be small compared with either.

We next write in equation (68) $\alpha = -h$, $\alpha_n = -k$, $x = y/hk$. Let $h$ and $k$ increase indefinitely, and finally suppose the modulus of $y$ large. On multiplying the equation by $\Pi(h)\ \Pi(k)$ the limiting form of the left-hand member, whatever be the modulus of $y$, is similar to (69) except that there is no $\alpha_n$, while the right-hand member considered as a limiting form of (70) becomes for any value of $y$

$$\frac{\Pi(h)}{\Pi(\rho + h - 1)} \int_{y/h}^{\infty} (y/h)^{1-\rho} (v - y/h)^{\rho+h-1} v^{-h-1} \chi(v)\, dv \ldots\ldots\ldots\ldots\ldots\ldots (78),$$

where $\chi(v)$ when $v$ is small is of the order of a power of $v$, and when $v$ is great is approximately equal to

$$\pi^{\frac{1}{2}} y^{\frac{1}{2}(\Sigma a - \sum_1^n \rho_r + \frac{1}{2})} e^{-2v^{\frac{1}{2}}},$$

provided the argument of $v$ lies between $-3\pi$ and $+3\pi$. When $h$ is increased indefinitely (78) becomes

$$y^{1-\rho} \int_{\epsilon y}^{\infty} v^{\rho-2} e^{-y/v} \chi(v)\, dv \quad \text{.....} \quad (79),$$

wherein, as may be seen by considering the integral of which this is a limiting form, the argument of every power of $v$ at the upper limit is zero; the argument of $y$ will be supposed to lie between $-3\pi$ and $+3\pi$ in order that the approximate form for $\chi(v)$ may be applicable.

By the substitutions $v = v'^2$, $y = y'^2$, $\epsilon = \epsilon'^2$, (79) is reduced to the same type as (72), $s$ having the value 2, and its evaluation thus leads to the result

$$2\pi \,.\, 3^{-\frac{1}{2}} y^{\frac{1}{2}(\Sigma a - \Sigma\rho + 1)} e^{-3y^{\frac{1}{3}}} \quad \text{.....} \quad (80),$$

$\Sigma\rho$ now including $\rho$.

The resulting equation is thus obvious; it is established for values of $y$ (or $x$) whose arguments lie between $-3\pi$ and $+3\pi$, and thus gives three independent relations. As in the preceding case the limits of argument may be extended to $-4\pi$ and $+4\pi$. This result is used in establishing the next case, and so on.

Thus we obtain the general result expressed by the following equation

$$\frac{\Pi(-\rho_1)\,\Pi(-\rho_2)\ldots\Pi(-\rho_n)}{\Pi(-\alpha_1)\,\Pi(-\alpha_2)\ldots\Pi(-\alpha_m)} F(\alpha_1, \alpha_2, \ldots \alpha_m;\ \rho_1, \rho_2, \ldots \rho_n;\ (-)^{n-m+1} x)$$

$$+ \frac{\Pi(\rho_1 - 2)\,\Pi(\rho_1 - \rho_2 - 1)\ldots\Pi(\rho_1 - \rho_n - 1)}{\Pi(\rho_1 - \alpha_1 - 1)\ldots\Pi(\rho_1 - \alpha_m - 1)} x^{1-\rho_1} F(\alpha_1 - \rho_1 + 1, \ldots \alpha_m - \rho_1 + 1;$$

$$2 - \rho_1, \ldots \rho_n - \rho_1 + 1;\ (-)^{n-m+1} x)$$

$+(n-1)$ other terms analogous to the last

$$\doteqdot (n+1-m)^{-\frac{1}{2}} (2\pi)^{\frac{n-m}{2}} v_1^{\frac{n-m}{2} + \Sigma a - \Sigma\rho} e^{-(n+1-m) v_1} \quad \text{.....} \quad (81),$$

where $v_1^{n+1-m} = x$ and the argument of $v_1$ may have any value between $-\pi$ and $+\pi$. This equation is thus equivalent to $n+1-m$ independent equations.

I have verified that if, by means of the linear equations thus obtained, one of the convergent series be expressed in terms of the divergent functions, the result obtained for a real positive $x$ agrees with that of Stokes (*loc. cit.*).

A complicated series may be obtained for the integral (72), by integrating by parts, writing $y^s = v_1^{s+1}$, $sv + v_1^{s+1} v^{-s} - (s+1) v_1 = u^2$, and expanding $v^{p+1}$ in powers of $u$ by Lagrange's theorem.

If we denote $z^{s/2}(1+2z+3z^2+\ldots sz^{s-1})^{-1/2}$ by $\phi(z)$ the result thus obtained is

$$\pi^{\frac{1}{2}} v_1^{p+\frac{1}{2}} e^{-(s+1)v_1} \left\{ \left(\frac{2}{s(s+1)}\right)^{1/2} + \left[\frac{d^2}{dz^2}(z^p\{\phi(z)\}^3)\right]_{z=1} \frac{1}{4v_1} \right.$$
$$\left. + \left[\frac{d^4}{dz^4}(z^p\{\phi(z)\}^5)\right]_{z=1} \frac{1}{1\,.\,2\,.\,(4v_1)^2} + \ldots \right\} \ldots\ldots\ldots\ldots(82).$$

Thus if there are no $\alpha$'s and $s$ $\rho$'s the right-hand member of (81), in order that the equation should be exact, should be increased in the ratio of the above series to its first term, $p$ being $(s-1)/2-\Sigma\rho$. If $s=1$ we may thus derive the semi-convergent series connected with the Bessel Functions. For other values the series is complicated.

If there are any $\alpha$'s the result is still more complicated.

In the case in which $m=n$, the result in case there is only one $\alpha$ and one $\rho$ is given in equation (27).

If another $\alpha$ and $\rho$ are now introduced it may be shown that to make equation (61) exact the right-hand member should be multiplied by the infinite series of divergent series:—

$$\phi(\rho_1-\alpha_2-\alpha_1+1,\ \rho_2-\alpha_2;\ -1/x) - \frac{(1-\alpha_1)(\rho_1-\alpha_1)}{1\,.\,x}\phi(\rho_1-\alpha_2-\alpha_1+2,\ \rho_2-\alpha_2;\ -1/x)$$
$$+\frac{(1-\alpha_1)(2-\alpha_1)(\rho_1-\alpha_1)(\rho_1-\alpha_1+1)}{1\,.\,2\,.\,x^2}\phi(\rho_1-\alpha_2-\alpha_1+3,\ \rho_2-\alpha_2;\ -1/x)+\ldots$$

Another $\alpha$ and $\rho$ make this series triply infinite, and so on.

If there is one $\alpha$ and two $\rho$'s the limiting form of the above shows that the right-hand member of (77) should in that case be multiplied by

$$\phi\left(\rho_1-\rho_2-\alpha+3/2,\ \alpha+\rho_2-\rho_1-1/2;\ \frac{-1}{2v_1}\right)$$
$$-\frac{(1-\alpha)(\rho_1-\alpha)}{1\,.\,v_1}\phi\left(\rho_1-\rho_2-\alpha+5/2,\ \alpha+\rho_2-\rho_1-3/2;\ \frac{-1}{2v_1}\right)$$
$$+\frac{(1-\alpha)(2-\alpha)(\rho_1-\alpha)(\rho_1-\alpha+1)}{1\,.\,2\,.\,v_1^2}\phi\left(\rho_1-\rho_2-\alpha+7/2,\ \alpha+\rho_2-\rho_1-5/2;\ \frac{-1}{2v_1}\right),$$

where $v_1^2=x$. Each series here is a semi-convergent series connected with the Bessel functions. Another $\alpha$ and $\rho$ make this series triply infinite, and so on.

It is to be noted that in the case in which $m=n+1$ no numerical connexion has been here established between the series which proceed in ascending and those which proceed in descending powers of $x$.

# INDEX TO VOL. XVII.


Cayley, on the product of two hypergeometric series, 1; on change of independent variables, 231
Characters, generic of quadratic forms, 135
CHREE, C., Sc.D., A method of solution of the equations of elasticity, 201
Classes of quadratic forms, 139
Currents, electric, decay of, in conducting spheroids, 104

Earth, the, equations of elasticity applied to, 210
Elasticity, a method of solution of the equations of, 201
Elliott, Prof., on Cyclicants and Reciprocants, 231, 267, 273, 279
Ellipsoids and cylinders, elastic, 201, 213, 223
Elliptic disc, rotating, elastic, 217
Equation, the study of the differential, $(\nabla^2+\kappa^2)\psi=0$, 41

Fermat's theorem for quadratic complex numbers, 119

GALLOP, E. G., On the change of a system of independent variables, 231
Gauss's law of composition of quadratic forms, 111
Graphs, for the partition of numbers, 151

Heat, equation of conduction of, 41, 105
Hypergeometric series, on the product of two, 1; on divergent, 171, 283

Kerr, experiments in magneto-optics, 30
Klein, Prof., lectures on ideal numbers, 109

Larmor, J., theory of the electric medium, 17
LEATHEM, J. G., On deducing magneto-optic phenomena from an energy function, 16

MACLAURIN, R. C., On the solutions of $(\nabla^2+\kappa^2)\psi=0$, 41
MACMAHON, Major P. A., Partitions of numbers whose graphs possess symmetry, 149

Magneto-optic phenomena deduced from an energy function, 16

Numbers, Partitions of, whose graphs possess symmetry, 149
—— certain systems of quadratic complex, 109

Optics, and magnetism, theory of, 16
ORR, Prof. W. M'F., On hypergeometric series, 1, 171, 283
Oscillations, electrical, in a homœoidal layer of dielectric, 96

Partitions, theory of, 149
Physical applications of the equation $(\nabla^2+\kappa^2)\psi=0$, 41
Pochhammer, on contour integrals, 173

Quadratic, complex numbers, 109; quadratic forms, 131
Quantics, algebra of, 249

Reciprocity, laws of quadratic, 124
Reflexion, magnetic, 25

Sylvester's constructive theory of partitions, 149; on change of independent variable, 231

Tables of quadratic complex numbers, 146

Variables, independent, change of, 231
Vibration, of an elliptic plate, 73; of electricity in a cylindrical cavity, 76; of spheroidal sheets of air, 91

Waves, reflexion and refraction of, at a plane surface of magnetised metal, 22
—— scattering of, by an obstructing spheroid, 101
WESTERN, A. E., Certain systems of quadratic complex numbers, 109

Zeeman, experiments in magneto-optics, 32

Cambridge:
PRINTED BY J. & C. F. CLAY,
AT THE UNIVERSITY PRESS.